UNIT III
DIFFERENTIAL CALCULUS

Centre of Curvature

$$\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2)$$

$$\bar{y} = y + \frac{1}{y_2}(1+y_1^2)$$

$(\bar{x}, \bar{y})$ is the Coordinate of the centre of curvature.

Circle of curvature :

The equation of the circle of curvature is

$$(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$$

Problems :

(1) Find the centre of curvature of $y = x^2$ at the origin.

Soln : $y = x^2$ Point : $(0,0)$

$$\frac{dy}{dx} = y_1 = 2x \quad ; \quad y_1(0,0) = 2(0) = 0$$

$$\frac{d^2y}{dx^2} = y_2 = 2 \quad ; \quad y_2(0,0) = 2$$

The centre of curvature is,

$$\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2) \qquad \bar{y} = y + \frac{1}{y_2}(1+y_1^2)$$

$$= 0 - \frac{0}{2}(1+0) \qquad = 0 + \frac{1}{2}(1+0)$$

$$\bar{x} = 0 \qquad \bar{y} = \frac{1}{2}$$

$$\therefore \boxed{(\bar{x}, \bar{y}) = (0, 1/2)}$$

(2) Find the equation of the circle of curvature of the rectangular hyperbola $xy = 12$ at the point $(3, 4)$.

Soln:

The equation of circle of curvature is given by,

$$(x - \bar{x})^2 + (y - \bar{y})^2 = \rho^2 \longrightarrow (1)$$

where,

$$\bar{x} = x - \frac{y_1}{y_2}(1 + y_1^2)$$

$$\bar{y} = y + \frac{1}{y_2}(1 + y_1^2)$$

$$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2}$$

Given: $xy = 12$ point: $(3, 4)$

$$x \frac{dy}{dx} + y(1) = 0$$

$$x y_1 + y = 0$$

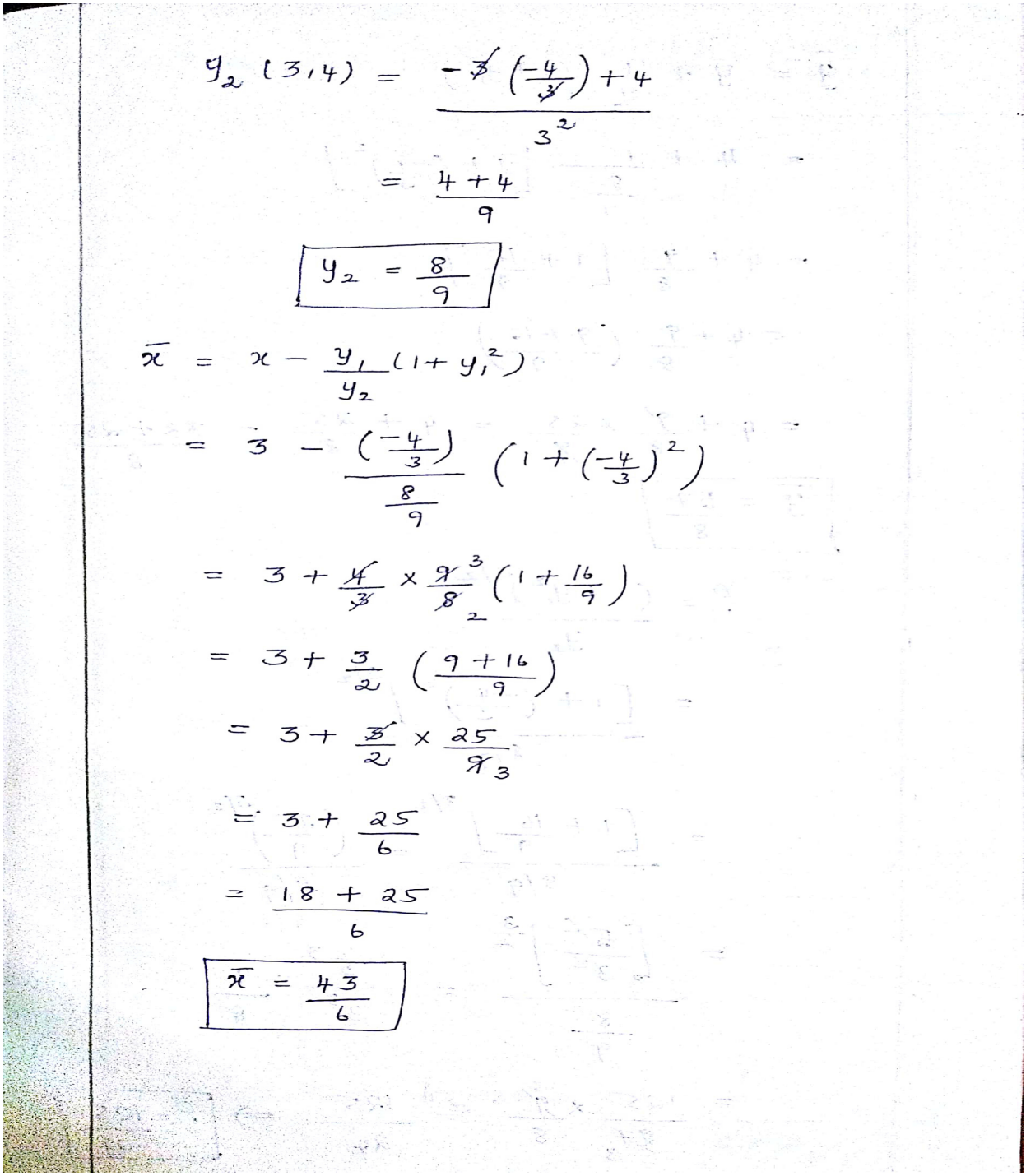
$$x y_1 = -y$$

$$y_1 = \frac{-y}{x}$$

$$\boxed{y_1(3,4) = \frac{-4}{3}}$$

$$y_2 = \frac{x(-y_1) - (-y)(1)}{x^2}$$

$$= \frac{-xy_1 + y}{x^2}$$



$$y_2(3,4) = \frac{-3\left(\frac{-4}{3}\right)+4}{3^2}$$

$$= \frac{4+4}{9}$$

$$\boxed{y_2 = \frac{8}{9}}$$

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= 3 - \frac{\left(\frac{-4}{3}\right)}{\frac{8}{9}}\left(1+\left(\frac{-4}{3}\right)^2\right)$$

$$= 3 + \frac{4}{3} \times \frac{9^3}{8_2}\left(1+\frac{16}{9}\right)$$

$$= 3 + \frac{3}{2}\left(\frac{9+16}{9}\right)$$

$$= 3 + \frac{3}{2} \times \frac{25}{9_3}$$

$$= 3 + \frac{25}{6}$$

$$= \frac{18+25}{6}$$

$$\boxed{\bar{x} = \frac{43}{6}}$$

$$\bar{y} = y + \frac{1}{y_2}\left(1+y_1^2\right)$$

$$= 4 + \frac{1}{\frac{8}{9}}\left[1+\left(\frac{-4}{3}\right)^2\right]$$

$$= 4 + \frac{9}{8}\left[1+\frac{16}{9}\right]$$

$$= 4 + \frac{9}{8}\left(\frac{9+16}{9}\right)$$

$$= 4 + \frac{\cancel{9}}{8} \times \frac{25}{\cancel{9}} = 4 + \frac{25}{8} = \frac{32+25}{8}$$

$$\boxed{\bar{y} = \frac{57}{8}}$$

$$\rho = \frac{\left(1+y_1^2\right)^{3/2}}{y_2}$$

$$= \frac{\left[1+\left(\frac{-4}{3}\right)^2\right]^{3/2}}{8/9}$$

$$= \frac{\left[1+\frac{16}{9}\right]^{3/2}}{8/9} = \frac{\left(\frac{25}{9}\right)^{3/2}}{8/9}$$

$$= \frac{\left[\frac{5^{\cancel{2}}}{3^{\cancel{2}}}\right]^{\frac{3}{\cancel{2}}}}{\frac{8}{9}} = \frac{5^3}{3^3} \times \frac{9}{8}$$

$$= \frac{125}{\cancel{27}_3} \times \frac{\cancel{9}}{8} = \frac{125}{24} \Rightarrow \boxed{\rho = \frac{125}{24}}$$

$$\therefore (1) \Rightarrow \left(x - \frac{43}{6}\right)^2 + \left(y - \frac{57}{8}\right)^2 = \left(\frac{125}{24}\right)^2$$