23MAT101 - MATRICES AND CALCULUS
UNIT-II ORTHOGONAL TRANSFORMATION OF A REAL SYMMETRIC MATRIX

Canonical form:

For a quadratic form $Q = X^T A X$, the canonical form is $Y^T D Y$ (or) $\lambda_1 Y_1^2 + \lambda_2 Y_2^2 + \dots + \lambda_n Y_n^2$

Problems:

① Determine the nature of the following quadratic form $x_1^2 + 2x_2^2$.

Soln:

Matrix of the quadratic form

$$= \begin{pmatrix} \text{coef of } x_1^2 & \frac{1}{2}\text{coef of } x_1x_2 & \frac{1}{2}\text{coef of } x_1x_3 \\ \frac{1}{2}\text{coef of } x_2x_1 & \text{coef of } x_2^2 & \frac{1}{2}\text{coef of } x_2x_3 \\ \frac{1}{2}\text{coef of } x_3x_1 & \frac{1}{2}\text{coef of } x_3x_2 & \text{coef of } x_3^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The eigen values are 1, 2, 0

Nature = positive semi definite.

(2) Find the rank, index, signature & nature for the following canonical forms:

(i) $y_1^2 + 3y_2^2 + 4y_3^2$

(ii) $-y_1^2 + y_2^2 + 4y_3^2$

(iii) $3y_2^2 + 15y_3^2$

Soln:

(i) $y_1^2 + 3y_2^2 + 4y_3^2$

Rank = number of non-zero eigen values.

= 3

Index = number of positive square terms

= 3

Signature = Difference between positive & negative square terms.

= 3 - 0 = 3

Nature = positive definite.

(ii) $-y_1^2 + y_2^2 + 4y_3^2$

rank = 3

index = 2

signature = 2 - 1 = 2

Nature = Indefinite

(iii) $3y_2^2 + 15y_3^2$

$= 0y_1^2 + 3y_2^2 + 15y_3^2$

rank = 2
index = 2
signature = 2

Nature = positive semi definite.

(1) $2x^2 + 3y^2 + 2z^2 + 2xy$

$$= \begin{array}{c} \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \end{array}$$

(2) $2x_1^2 + x_2^2 - 3x_3^2 + 12x_1x_2 - 8x_2x_3 - 4x_1x_3$

$$= \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 2 & 6 & -2 \\ 6 & 1 & -4 \\ -2 & -4 & -3 \end{bmatrix} \end{array}$$