UNIT - III

DIFFERENTIAL CALCULUS

I. Basic differentiation formulas:

S.No	y	dy/dx
1	k (Constant)	0
2	x^n	$n x^{n-1}$
3	e^{ax}	$a e^{ax}$
4	$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
5	$\log x$	$\frac{1}{x}$
6	a^x	$a^x \log a$
7	$\sin x$	$\cos x$
8	$\cos x$	$-\sin x$
9	$\tan x$	$\sec^2 x$
10	$\cot x$	$-\operatorname{cosec}^2 x$
11	$\sec x$	$\sec x \tan x$
12	$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$.

S.No	y	dy/dx
(13)	$\sin hx$	$\cos hx$
(14)	$\cos hx$	$\sin hx$.

II. <u>Product rule</u>:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} = uv' + vu'$$

where u & v are functions of x.

III. <u>Quotient rule</u>:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{vu' - uv'}{v^2}$$

IV. If $x = x(t)$, $y = y(t)$ where 't' is a parameter then,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{dy}{dx}\right)\frac{dt}{dt}$$

$$= \frac{d}{dx}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$$

$$= \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{1}{dx/dt}$$

Curvature of a curve:

The rate of bending of a curve in any interval is called the curvature of the curve in that interval. It is denoted by k.

Note: The curvature of a straight line is zero.

Radius of curvature:

It is defined as the reciprocal of the curvature of the curve and is denoted by ρ.

$$\boxed{\rho = \frac{1}{k}}$$

Note:

The radius of curvature at every point of the circle is equal to the radius of the circle i.e $\rho = r$.

Hence the curvature is, $k = \frac{1}{\rho} = \frac{1}{r}$

Formula for radius of curvature:

$$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2} \quad (\text{or}) \quad \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2}$$

where $y_1 = \frac{dy}{dx}$, $y_2 = \frac{d^2y}{dx^2}$

Note: When $\frac{dy}{dx}$ becomes ∞, $\rho = \frac{\left[1 + \left(\frac{dx}{dy}\right)^2\right]^{3/2}}{d^2x/dy^2}$

is the alternative formula for radius of curvature.

Problems:

① Find the radius of curvature at any point of the catenary $y = c\cosh(x/c)$

Soln:

$$y = c\cosh(x/c) \longrightarrow ①$$

$$y_1 = c\sinh(x/c)\frac{1}{c} = \sinh(x/c)$$

$$y_2 = \cosh(x/c)\frac{1}{c}$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$= \frac{[1+\sinh^2(x/c)]^{3/2}}{\cosh(x/c)} \cdot c$$

$[\cosh^2x - \sinh^2x$

$$= \frac{[\cosh^2(x/c)]^{3/2}}{\cosh(x/c)} \cdot c$$

$$\rho = c\cosh^2(x/c) \longrightarrow ②$$

From ①, $\cosh(x/c) = y/c$

Subs this in ②, we get

$$\rho = c \cdot \frac{y^2}{c^2}$$

$$\boxed{\rho = y^2/c}$$

② Find ρ for (i) $y = \log \sin x$ (ii) $y = c \log \sec(x/c)$

Soln:

(i) $y = \log \sin x$

$y_1 = \frac{1}{\sin x} \cos x = \cot x.$

$y_2 = -\operatorname{cosec}^2 x$

$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+\cot^2 x)^{3/2}}{-\operatorname{cosec}^2 x}$

$= \frac{\operatorname{cosec}^3 x}{-\operatorname{cosec}^2 x} = -\operatorname{cosec} x$

$|\rho| = \operatorname{cosec} x.$

(ii) $y = c \log(\sec(x/c))$

$y_1 = c \frac{1}{\sec(x/c)} \sec\left(\frac{x}{c}\right) \tan\left(\frac{x}{c}\right) \cdot \frac{1}{c}$

$y_1 = \tan\left(\frac{x}{c}\right)$

$y_2 = \frac{1}{c} \sec^2\left(\frac{x}{c}\right)$

$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{\left[1+\tan^2(x/c)\right]^{3/2}}{\frac{1}{c}\sec^2(x/c)}$

$= c\frac{\sec^3(x/c)}{\sec^2(x/c)} \Rightarrow \rho = c \sec(x/c)$

③ Find the radius of curvature for the curves.

(i) $y = e^x$, where it crosses the y-axis

(ii) $y^2 = x^2 + 8$ at the point $(-2, 0)$

(iii) $xy = c^2$ at $x = c$

(iv) $y^2 = \frac{a^3 - x^3}{x}$ at $(a, 0)$

(v) $y = \frac{\log x}{x}$ at $x = 1$

(vi) $x^3 + y^3 = 3axy$ at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

Soln:

(i) $y = e^x$

on y-axis $x = 0$

when $x = 0$, $y = e^0 = 1$ $\therefore y = 1$

$\therefore$ The point is $(0, 1)$

$y = e^x$

$y_1 = e^x$; y_1 at $(0,1) = 1$

$y_2 = e^x$; y_2 at $(0,1) = 1$

$\therefore \rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+1)^{3/2}}{1} = 2^{3/2}$

$$\boxed{\rho = 2\sqrt{2}}$$

(ii) $y^2 = x^3 + 8$

Diff w.r.t 'x', we get

$2yy_1 = 3x^2$

$\therefore y_1 = \frac{3x^2}{2y}$; $y_1(-2, 0) = \infty$

$\therefore \frac{dx}{dy} = 0$; $\frac{dx}{dy} = \frac{2y}{3x^2}$

$$\frac{d^2x}{dy^2} = \frac{(3x^2)2 - (2y)6x\,dx/dy}{9x^4}$$

$$\left(\frac{d^2x}{dy^2}\right)_{(-2,0)} = \frac{6 \times 4}{9 \times 16} = \frac{1}{6}$$

$$\rho = \frac{\left[1 + \left(\frac{dx}{dy}\right)^2\right]^{3/2}}{d^2x/dy^2} = \frac{(1+0)^{3/2}}{1/6}$$

$$\boxed{\rho = 6}$$

(iii) $xy = c^2$ at $x = c$

If $x = c$, $xy = c^2$ will be

$$y = c^2/x = c^2/c = c$$

$$\boxed{y = c}$$

$\therefore$ The point is (c, c)

$$xy = c^2$$

$$xy_1 + y = 0$$

$$y_1 = -y/x$$

$$y_1 \text{ at } (c,c) \text{ is } = -c/c = -1$$

$$y_2 = \frac{x\left(-dy/dx\right) - (-y)\cdot 1}{x^2}$$

$$= \frac{-x \times (-y/x) + y}{x^2} = \frac{2y}{x^2}$$

$$y_2 \text{ at } (c,c) = \frac{2c}{c^2} = \frac{2}{c}$$

$$\therefore \rho = \frac{(1 + y_1^2)^{3/2}}{y_2}$$

$$= \frac{\left[1 + (-1)^2\right]^{3/2}}{2/c} = \frac{2^{3/2}c}{2}$$

$$= \frac{2\sqrt{2}\,c}{2}$$

$$\boxed{P = \sqrt{2}\cdot c}$$

(iv) $y^2 = \frac{a^3 - x^3}{x}$ at $(a, 0)$

$$2yy_1 = \frac{-a^3}{x^2} - 2x$$

$$y_1 = \frac{-a^3}{2yx^2} - \frac{x}{y} = \frac{-a^3 - 2x^3}{2x^2 y}$$

$$y_1 \text{ at } (a,0) = \infty$$

$$\therefore \frac{dx}{dy} = 0 \quad ; \frac{dx}{dy} = \frac{-2x^2y}{a^3 + 2x^3}$$

$$\frac{d^2x}{dy^2} = \frac{(a^3+2x^3)(-2)\left(x^2 + 2x\frac{dy}{dx}\cdot y\right) + 2x^2y\left(6x^2\frac{dx}{dy}\right)}{(a^3+2x^3)^2}$$

$$\left(\frac{d^2x}{dy^2}\right)_{(a,0)} = \frac{-2(3a^3)a^2}{9a^6} = \frac{-2}{3a}$$

$$\rho = \frac{\left[1 + \left(\frac{dx}{dy}\right)^2\right]^{3/2}}{d^2x/dy^2} = \frac{1}{-2/3a}$$

$$\rho = \frac{-3a}{2}$$

$$\boxed{|\rho| = \frac{3a}{2}}$$

(v) $y = \frac{\log x}{x}$ at $x = 1$

$$y = \frac{1 - \log x}{x^4}$$

when $x = 1$, $y = 0$
$\therefore (1, 0)$ is the point.

$$y_1 (1,0) = 1$$

$$y_2 = \frac{x^2 (1/x) - (1 - \log x) 2x}{x^4}$$

$$= \frac{-x - (1 - \log x) 2x}{x^4} = \frac{-1 - (1 - \log x) 2}{x^3}$$

$$y_2 (1,0) = -1 - 2 = -3$$

$$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2} = \frac{(2)^{3/2}}{-3} = \frac{-2\sqrt{2}}{3}$$

$$|\rho| = \frac{2\sqrt{2}}{3}$$

(vi) $x^3 + y^3 = 3axy$ at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

$$3x^2 + 3y^2 y_1 = 3a (xy_1 + y)$$

$$3y_1 (y^2 - ax) = 3 (ay - x^2)$$

$$y_1 = \frac{ay - x^2}{y^2 - ax}$$

$$y_1 \text{ at } \left(\frac{3a}{2}, \frac{3a}{2}\right) = \frac{a \times \frac{3a}{2} - \frac{9a^2}{4}}{\frac{9a^2}{4} - a \times \frac{3a}{2}}$$

$$= \frac{-\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)}{\frac{9a^2}{4} - \frac{3a^2}{2}} = -1$$

$$y_2 = \frac{(y^2 - ax)(ay_1 - 2x) - (ay - x^2)(2yy_1 - a)}{(y^2 - ax)^2}$$

$$y_2\left(\frac{3a}{2}, \frac{3a}{2}\right) = \frac{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)(-a - 3a) - \left(\frac{3a^2}{2} - \frac{9a^2}{4}\right)(-3a - a)}{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)^2}$$

$$= \frac{-4a - 4a}{3a^2/4} = \frac{-8a \times 4}{3a^2}$$

$$y_2 = \frac{-32}{3a}$$

$$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2} = \frac{(1+1)^{3/2}}{-32/3a}$$

$$= \frac{2\sqrt{2} \times 3a}{-32} = \frac{-3\sqrt{2}\,a}{16}$$

$$|\rho| = \frac{3\sqrt{2}\,a}{16}$$