UNIT-1 - MATRICES

Introduction:

The term matrix was apparently coined by Sylvester about 1850, but was introduced first by Cayley in 1860. By a 'matrix' we mean an "arrangement" or "rectangular array" of numbers. Matrices applications are finding the solution of system of linear equations, probability, mathematical economics, quantum mechanics, electrical networks, curve fitting, transportation problems, frameworks in mechanics.

Definition:

A set of 'mn' numbers arranged in a rectangular array having 'm' rows and 'n' columns, the numbers being enclosed by brackets [ ] or ( ), is called an $m \times n$ matrix.

Each of the 'mn' numbers is called an element of the matrix.

An $m \times n$ matrix is usually written as

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

or $A = [a_{ij}]_{m \times n}$

or $A = [a_{ij}]$, where $i = 1, 2, \ldots m$, $j = 1, 2, \ldots n$.

Real Matrix:

A matrix is said to be real if all its elements are real numbers.

ex: $\begin{bmatrix} \sqrt{5} & -3 & 1 \\ 0 & -\sqrt{2} & 7 \end{bmatrix}$ is a real matrix.

Square matrix:

A matrix in which the number of rows is equal to the number of columns is called a square matrix, otherwise, it is said to be a rectangular matrix.

Thus, a matrix $A = [a_{ij}]_{m \times n}$ is a square matrix if $m = n$ and a rectangular matrix if $m \neq n$.

ex: $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

The elements, a_{11}, a_{22}, a_{33} of a square matrix are called its diagonal ~~matrix~~ elements and the diagonal along which these elements lie is called the principal diagonal.

The sum of the diagonal elements of a square matrix is called its trace.

Row matrix:

A matrix having only one row and any number of columns. (ie $1 \times n$ matrix).

ex: $\begin{bmatrix} 2 & 5 & -3 & 0 \end{bmatrix}$

Column matrix:

A matrix having only one column and any number of rows. [(ii) $m \times 1$ matrix]

ex: $\begin{bmatrix} \sqrt{2} \\ 0 \\ -1 \end{bmatrix}$

Null Matrix:

A matrix in which each element is zero is called a null matrix or a zero matrix and it is denoted by $O_{m\times n}$.

ex:

$$O_{3\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad O_{2\times 4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Sub Matrix

A matrix obtained from a given matrix A by deleting some of its rows or column or both is called a sub-matrix of A.

ex:

$$B = \begin{bmatrix} 3 & 0 \\ 1 & 4 \end{bmatrix} \text{ is a sub-matrix of } A = \begin{bmatrix} 0 & -1 & 2 & 5 \\ 3 & 5 & 0 & 7 \\ 1 & 6 & 4 & -2 \end{bmatrix}$$

Diagonal Matrix:

A square matrix in which all non-diagonal elements are zero is called a diagonal matrix.

(or) $A = [a_{ij}]_{m\times n}$ is a diagonal matrix if $a_{ij} = 0$, for $i \neq j$.

ex:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Scalar Matrix:

A square matrix in which all diagonal elements are equal to a scalar, say k, is called a scalar matrix.

ex:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Unit Matrix or Identity Matrix

A square matrix in which each diagonal element is unity (ie) 1) is called a unit matrix or identity matrix.

ex:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$$

Upper Triangular Matrix:

A square matrix in which all the elements below the principal diagonal are zero is called an upper triangular matrix.

ex:

$$\begin{bmatrix} 2 & 3 & 4 \\ 0 & -1 & 5 \\ 0 & 0 & 3 \end{bmatrix}$$

Lower Triangular Matrix:

A square matrix in which all the elements above the principal diagonal are zero is called a lower triangular matrix.

ex:

$$\begin{bmatrix} -1 & 0 & 0 \\ 5 & 6 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

Triangular Matrix

A square matrix in which all the elements either below or above the principal diagonal are zero is called a triangular matrix. Thus, a triangular matrix is either upper triangular or lower triangular.

Equal Matrices

Two matrices A and B are said to be equal if and only if they have same order and their corresponding elements are equal.

Addition and Subtraction of Matrices

$$A = \begin{bmatrix} 2 & 5 & -1 \\ 3 & 0 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & -6 & 2 \\ -2 & 5 & 7 \end{bmatrix}$$

$$A+B = \begin{bmatrix} 3 & -1 & 1 \\ 1 & 5 & 11 \end{bmatrix}, \quad A-B = \begin{bmatrix} 1 & 11 & -3 \\ 5 & -5 & -3 \end{bmatrix}.$$

Matrix Multiplication

Two matrices A and B are said to be Conformal for multiplication if the number of columns of A is equal to the number of rows of B.

Scalar Multiplication

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \Rightarrow KA = \begin{bmatrix} Ka_1 & Ka_2 & Ka_3 \\ Kb_1 & Kb_2 & Kb_3 \end{bmatrix}$$

Transpose of a Matrix

The matrix obtained from A by changing its rows into columns and columns into rows is called the transpose of A and is denoted by A' or A^T.

ex:

$$A = \begin{bmatrix} 1 & 0 & 2 & 5 \\ 2 & -1 & 3 & 7 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 2 & 3 \\ 5 & 7 \end{bmatrix}$$

Symmetric Matrix

A square matrix $A = [a_{ij}]$ is said to be Symmetric if $A' = A$.

ex: $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 4 \end{bmatrix}$, $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

Skew-Symmetric Matrix or Anti-symmetric Matrix

A square matrix $A = [a_{ij}]$ is said to be skew-symmetric if $A = -A$.

ex: $\begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 1 \\ 3 & -1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & h & -g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}$

Orthogonal matrix

A square matrix A is called an orthogonal matrix if $AA^T = I = A^TA$.

ex: $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

Singular and non-Singular Matrices

A square matrix A is said to be singular if $|A| = 0$ and non-singular if $|A| \neq 0$.

Inverse of a Square Matrix

Let A and B be any two square matrices such that $AB = BA = I$. Then the matrix A is said to be invertible and B is called the inverse of A.

Note:

Only square matrices can be invertible.