1. Find the eigen value and eigen vector of the matrix $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}$ Symmetric matrix with non Repeated eigen values.

Let $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}$

Step-I :- To find the charac eqn

The characteristic equation is given by

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0.$$

S_1 = Sum of diagonal elements

$= 2+1+1 = 4$

$$S_2 = \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$$

$= (1-4) + (2-1) + 2(-1)$

$= -3 + 1 + 1$

$S_2 = -1$

$S_3 = |A| = 2(-3) - 1(-1) - 1(-1)$

$= -6 + 1 + 1 = -4$

$\boxed{S_3 = -4}$.

Step: 2. ∴ The characteristic equation is $\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$

To solve charac eqn :-

	1	−4	−1	4
1		1	−3	−4
	1	−3	−4	0

$\lambda^2 - 3\lambda - 4 = 0$

$(\lambda^2 - 4\lambda + \lambda - 4) = 0$.

$(\lambda - 4)(\lambda + 1) = 0$.

$\Rightarrow$ Eigen values are $-1, 1, 4$

Step : 3

To find eigen vectors :-

$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ satisfies the equation

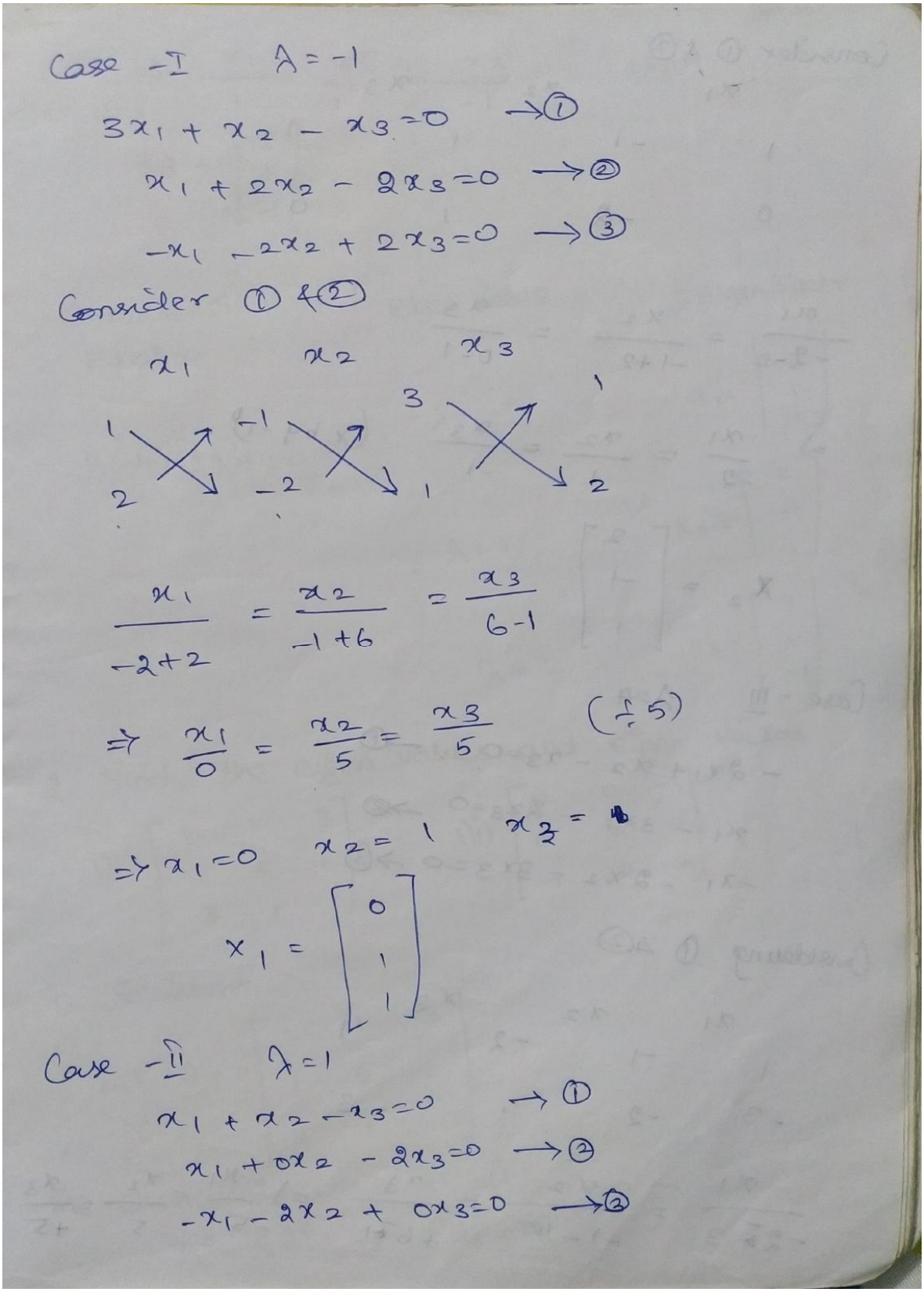
$(A - \lambda I)X = 0$

$$\begin{bmatrix} 2-\lambda & 1 & -1 \\ 1 & 1-\lambda & -2 \\ 1 & -2 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

$(2-\lambda)x_1 + x_2 - x_3 = 0$.

$+x_1 + (1-\lambda)x_2 - 2x_3 = 0$

$-x_1 - 2x_2 + (1-\lambda)x_3 = 0$


Case -I λ = -1
3x1 + x2 − x3 = 0 → (1)
x1 + 2x2 − 2x3 = 0 → (2)
−x1 − 2x2 + 2x3 = 0 → (3)
Consider (1) & (2)
x1 x2 x3
1 −1 3 1
2 −2 1 2
x1/(−2+2) = x2/(−1+6) = x3/(6−1)
⇒ x1/0 = x2/5 = x3/5 (÷5)
⇒ x1 = 0 x2 = 1 x3 = 1
x1 = [0 1 1]
Case -II λ = 1
x1 + x2 − x3 = 0 → (1)
x1 + 0x2 − 2x3 = 0 → (2)
−x1 − 2x2 + 0x3 = 0 → (2)

Consider ① & ②

x_1	x_2	x_3	
1	-1	1	1
0	-2	1	0

$$\frac{x_1}{-2-0} = \frac{x_2}{-1+2} = \frac{x_3}{0-1}$$

$$\Rightarrow \frac{x_1}{-2} = \frac{x_2}{1} = \frac{x_3}{-1} \quad (\times \text{by } -1)$$

$$X_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Case - III $\quad \lambda = 4$

$$-2x_1 + x_2 - x_3 = 0 \quad \rightarrow ①$$

$$x_1 - 3x_2 - 2x_3 = 0 \quad \rightarrow ②$$

$$-x_1 - 2x_2 - 3x_3 = 0 \quad \rightarrow ③$$

Considering ① & ②

x_1	x_2	x_3	
1	-1	-2	1
-3	-2	1	-3

$$\frac{x_1}{-2+3} = \frac{x_2}{-1-4} = \frac{x_3}{+6+1} \Rightarrow \frac{x_1}{-5} = \frac{x_2}{-5} = \frac{x_3}{+5}$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{-1}$$

$$X_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Characteristic Equation	Eigen Value	Eigen Vector
$\lambda^2 - 4\lambda^2 + \lambda + 4 = 0$	$\lambda = -1$	$X_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$
	$\lambda = 1$	$X_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$
	$\lambda = 4$	$X_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

2. Find the eigen values and eigen vectors

i) $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$

i) Solution:-

Let $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

The characteristic equation is

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0.$$

$(A-\lambda I)X = 0$. $\quad \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

$$(1-\lambda)x_1 + x_2 + 3x_3 = 0$$
$$x_1 + (5-\lambda)x_2 + x_3 = 0.$$
$$3x_1 + x_2 + (1-\lambda)x_3 = 0.$$

Case -I when $\lambda = -2$

$$3x_1 + x_2 + 3x_3 = 0 \quad \rightarrow (1)$$
$$x_1 + 7x_2 + x_3 = 0 \quad \rightarrow (2)$$
$$3x_1 + x_2 + 3x_3 = 0 \quad \rightarrow (3)$$

Consider (1) & (2)

x_1	x_2	x_3	x_1
1	3	3	1
7	1	1	7

$$\frac{x_1}{-20} = \frac{x_2}{0} = \frac{x_3}{20}$$

$$\frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1} \Rightarrow X_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \text{ or } \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

S_1 = Sum of the diagonal elements

$= 1+5+1$

$= 7$

S_2 = Sum of the minors of diagonal elements

$$= \begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix}$$

$= (5-1) + (1-9) + (5-1)$

$= 8 - 8 = 0.$

$S_3 = |A| = 1(4) - 1(-2) + 3(-14)$

$= 4 + 2 - 42$

$= -36.$

∴ The characteristic equation is

$\lambda^3 - 7\lambda^2 + 36 = 0.$

	1	−7	0	36
−2	↓	−2	18	−36
	1	−9	18	0

$\lambda^2 - 9\lambda + 18 = 0. \Rightarrow \lambda^2 - 6\lambda - 3\lambda + 18 = 0.$

$\lambda = 6, 3, -2$

∴ Eigen values are $-2, 3, 6$

$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ satisfies the equation.

Case II $\lambda = 3$

$$-2x_1 + x_2 + 3x_3 = 0 \rightarrow (1)$$
$$x_1 + 2x_2 + x_3 = 0 \rightarrow (2)$$
$$3x_1 + x_2 - 2x_3 = 0 \rightarrow (3)$$

Consider (1) & (2)

x_1	x_2	x_3	
1	3	-2	1
2	1	1	2

$$\frac{x_1}{1-6} = \frac{x_2}{3+2} = \frac{x_3}{-4-1} \Rightarrow \frac{x_1}{-5} = \frac{x_2}{5} = \frac{x_3}{-5}$$

$$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1} \quad \therefore X_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Case - III When $\lambda = 6$

$$-5x_1 + x_2 + 3x_3 = 0 \rightarrow (1)$$
$$x_1 - x_2 + x_3 = 0 \rightarrow (2)$$
$$3x_1 + x_2 - 5x_3 = 0 \rightarrow (3)$$

Consider (1) & (2)

x_1	x_2	x_3	
1	3	-5	1
-1	1	1	-1

$$\frac{x_1}{1+3} = \frac{x_2}{3+5} = \frac{x_3}{5-1}$$

$$\frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{4}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1} \quad \therefore X_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

— x —

ii) Let $A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$

The characteristic equation is

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0.$$

S_1 = Sum of the diagonal elements

$= 2+2+2 = 6.$

S_2 = Sum of the minors of the diagonal elements

$$= \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix}$$

$= 4 + 3 + 4 = 11$

$S_3 = |A| = 2(4) - 0(0) - 1(0+2)$

$= 8 - 2 = 6.$

∴ The characteristic equation is

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0.$$

	1	-6	11	-6
1	↓	1	-5	6
	1	-5	6	0

$\therefore \lambda^2 - 5\lambda + 6 = 0$

$\Rightarrow \lambda^2 - 3\lambda - 2\lambda + 6 = 0$

$\Rightarrow (\lambda - 3)(\lambda - 2) = 0$

$\therefore$ Eigen values are 1, 2, 3

$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ satisfies the equation $(A - \lambda I)X = 0$

$$\begin{bmatrix} 2-\lambda & 0 & -1 \\ 0 & 2-\lambda & 0 \\ -1 & 0 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

$(2-\lambda)x_1 + x_3 = 0.$

$(2-\lambda)x_2 = 0.$

$-x_1 + (2-\lambda)x_3 = 0$

Case I. When $\lambda = 1$

$x_1 + x_3 = 0$

$x_2 = 0$

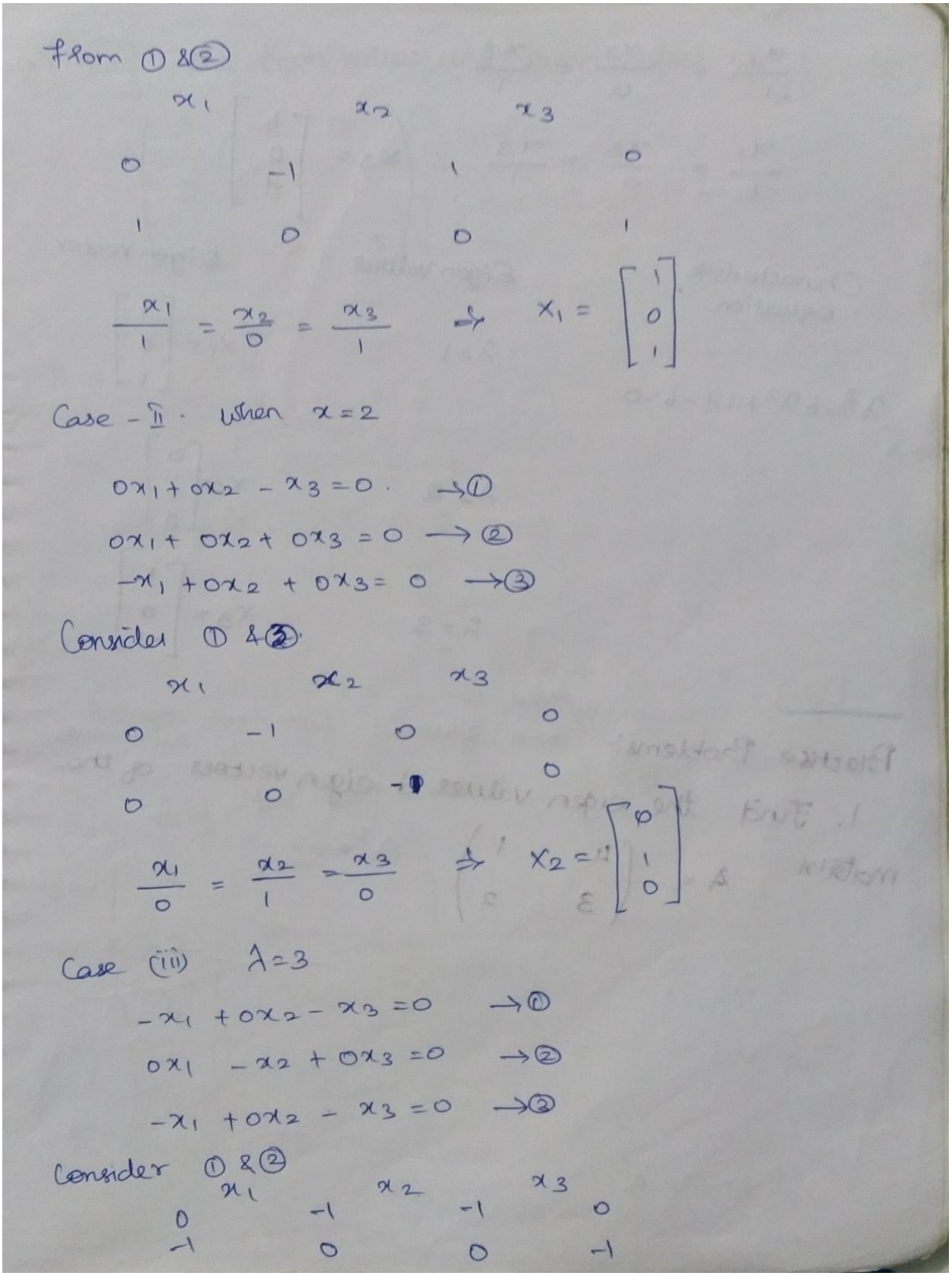
$-x_1 + x_3 = 0.$

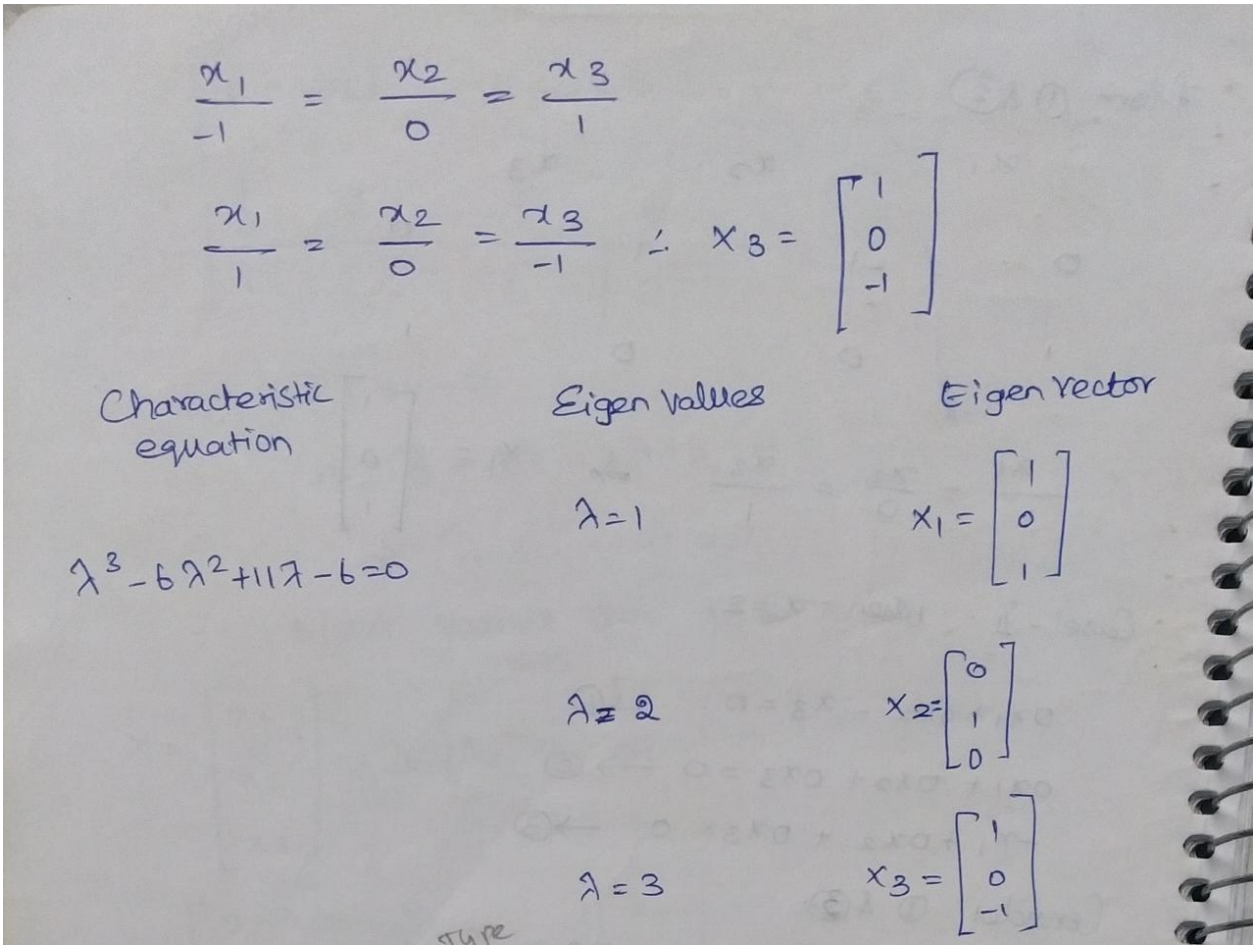
(or)

$x_1 + 0x_2 + x_3 = 0 \rightarrow (1)$

$0x_1 + x_2 + 0x_3 = 0 \rightarrow (2)$

$-x_1 + 0x_2 + x_3 = 0 \rightarrow (3)$


from ① & ②
x_1 x_2 x_3
0 −1 1 0
1 0 0 1
$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{1}$ ⇒ $X_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$
Case - II · When $\lambda = 2$
$0x_1 + 0x_2 - x_3 = 0$ → ①
$0x_1 + 0x_2 + 0x_3 = 0$ → ②
$-x_1 + 0x_2 + 0x_3 = 0$ → ③
Consider ① & ②
x_1 x_2 x_3
0 −1 0 0
0 0 −1 0
$\frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{0}$ ⇒ $X_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$
Case (iii) $\lambda = 3$
$-x_1 + 0x_2 - x_3 = 0$ → ①
$0x_1 - x_2 + 0x_3 = 0$ → ②
$-x_1 + 0x_2 - x_3 = 0$ → ③
Consider ① & ②
x_1 x_2 x_3
0 −1 −1 0
−1 0 0 −1



$$\frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1}$$

$$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{-1} \quad \therefore X_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$X_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Problems based on Non Symmetric Matrix with Repeated Eigen Values.

$$\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$$

Step 1:-

$S_1 = -2+1+0 = -1$

$S_2 = \begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & \\ 2 & \end{vmatrix}$

$= (0-12) + (0-3) + (-2-4)$

$= -21$

$S_3 = -2(-12) - 2(0-6) - 3(-4+1)$

$= 24+12+9 = 45$

$\therefore$ The characteristic eqn is $\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$.

Step : 2.

$\lambda = 5, -3, -3.$

Step 3:- To find the Eigenvectors

To find Eigen vectors solve $(A - \lambda I)X = 0$

$$\begin{bmatrix} (-2-\lambda) & 2 & -3 \\ 2 & (1-\lambda) & -6 \\ -1 & -2 & (0-\lambda) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

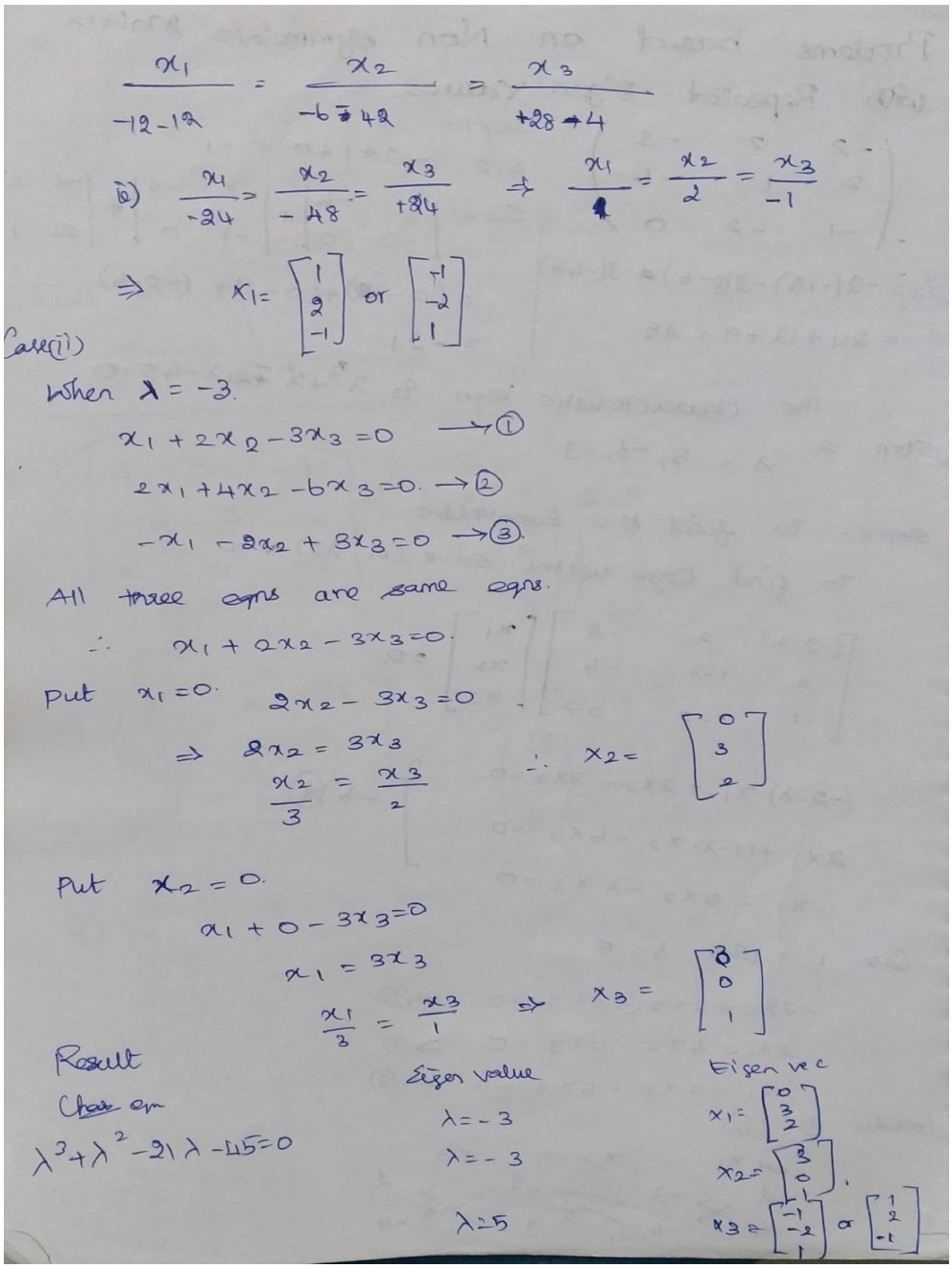
$$\left.\begin{aligned} (-2-\lambda)x_1 + 2x_2 - 3x_3 &= 0. \\ 2x_1 + (1-\lambda)x_2 - 6x_3 &= 0. \\ -x_1 - 2x_2 - \lambda x_3 &= 0 \end{aligned}\right\} \rightarrow (*)$$

Case 1) when $\lambda = 5$

$$-7x_1 + 2x_2 - 3x_3 = 0 \rightarrow (1)$$
$$2x_1 - 4x_2 - 6x_3 = 0 \rightarrow (2)$$
$$-x_1 - 2x_2 - 5x_3 = 0 \rightarrow (3)$$

Consider (1) & (2)

x_1	x_2	x_3
2	-3	-7
-4	-6	2


Case(i)
When λ = -3.
x1 + 2x2 - 3x3 = 0 → ①
2x1 + 4x2 - 6x3 = 0 → ②
-x1 - 2x2 + 3x3 = 0 → ③
All three eqns are same eqns.
Put x1 = 0.
Put x2 = 0.
Result
Char eqn
λ^3 + λ^2 - 21λ - 45 = 0
Eigen value
λ = -3
λ = -3
λ = 5
Eigen vec