



Cayley- Hamilton Theorem:
Every square matrix satisfies its own characteristic equation.

Uses of Cayley- Hamilton Theorem:

- To calculate (i) the positive integral powers of A
(ii) the inverse of a non-singular square matrix A.

1. verify Cayley- Hamilton theorem for the matrix $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$

Soln.

$$\text{Let } A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

The characteristic eqn. is $\lambda^2 - D_1\lambda + D_2 = 0$

$$D_1 = 1+1 = 2$$

$$D_2 = |A| = \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} = 1+4 = 5$$

$$\therefore \lambda^2 - 2\lambda + 5 = 0$$

To prove: $A^2 - 2A + 5I = 0$

$$\text{Now } A^2 = A \cdot A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 4 & -3 \end{bmatrix}$$

$$\text{and } A^2 - 2A + 5I = \begin{bmatrix} -3 & -4 \\ 4 & -3 \end{bmatrix} - 2 \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} + 5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

\therefore The given matrix satisfies its own characteristic eqn.

Hence Cayley Hamilton theorem is verified.

2. verify Cayley Hamilton theorem for $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$, find A^7 and A^{-1} .





Soln.

$$A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

The characteristic eqn. is $\lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0$

$$D_1 = 2 + 2 + 2 = 6$$

$D_2 =$ Sum of minors of main diagonal e.e.s.

$$= (4-1) + (4-2) + (4-1)$$

$$= 3 + 2 + 3$$

$$= 8$$

$$D_3 = |A|$$

$$= 2(4-1) + 1(-2+1) + 2(1-2)$$

$$= 3$$

$$\therefore \lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0$$

To prove: $A^3 - 6A^2 + 8A - 3I = 0$

Now $A^2 = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}$

$$A^3 = A \times A^2 = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \times \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix}$$

and $A^3 - 6A^2 + 8A - 3I$

$$= \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix} - 6 \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} + 8 \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$- 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$





UNIT I – MATRIX EIGENVALUE PROBLEMS

Cayley Hamilton Theorem(Statement only)

Hence Cayley Hamilton theorem is verified.

To find A^{-1} :

$$A^3 - 6A^2 + 8A - 3I = 0$$

$$(xA^{-1}) \quad A^3 - 6A^2 + 8A - 3A^{-1} = 0$$

$$3A^{-1} = A^3 - 6A^2 + 8A$$

$$A^{-1} = \frac{1}{3} [A^3 - 6A^2 + 8A]$$

$$A^{-1} = \frac{1}{3} \left[\begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - 6 \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 8 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right]$$

$$= \frac{1}{3} \begin{bmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$$

To find A^4 :

$$(xA) \quad A^4 - 6A^3 + 8A^2 - 3A = 0$$

$$A^4 = 6A^3 - 8A^2 + 3A$$

$$= 6 \begin{bmatrix} 29 & -28 & 38 \end{bmatrix}$$

$$= 6 [6A^2 - 8A + 3I] - 8A^2 + 3A$$

$$= 36A^2 - 48A + 18I - 8A^2 + 3A$$

$$A^4 = 28A^2 - 45A + 18I$$

$$= 28 \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - 45 \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 18 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 124 & -123 & 162 \\ -95 & 96 & -123 \\ 95 & -95 & 124 \end{bmatrix}$$

