Properties of eigen values and eigen vectors.

1. The sum of the eigen values of a matrix is the sum of the principal diagonal elements i.e., Trace of A.
2. The product of the eigen values of a matrix A is equal to its determinant.
3. If $\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_n$ are the eigen values of a matrix A then $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots \frac{1}{\lambda_n}$ are the eigen values of A^{-1}
4. If $\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_n$ are the eigen values of A then $K\lambda_1, K\lambda_2, \ldots K\lambda_n$ are the eigen values of KA, where K is a scalar.
5. If $\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_n$ are the eigen values of the matrix A then $\lambda_1^m, \lambda_2^m, \ldots \lambda_n^m$ are the eigen values of the matrix A^m (m being +ve integer)
6. If $\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_n$ are the eigen of the matrix A then $\lambda_1 - K; \lambda_2 - K, \ldots \lambda_n - K$ are the eigen values of the matrix $(A-KI)$
7. Eigen Values of the matrices A and A^T are the same.
8. The eigen values of a real symmetric matrix are all real.
9. The eigen values of a unitary matrices are of unit modulus

10. If x_1 and x_2 are the eigen vectors corresponding to same eigen value λ then $ax_1 + bx_2$ (a & b are constants) is also an eigen vector of A corresponding to λ.

11. The eigen vector corresponding to distinct eigen values of a real symmetric matrix are orthogonal.

12. An eigen Vector cannot correspond to two different eigen values.

13. The eigen value of an orthogonal matrix have the absolute value 1.

14. Two eigen vectors x_1 and x_2 are called orthogonal vectors if $x_1^T x_2 = 0$.

Problems :

(1) Find the sum and product of the eigen values of the matrix

(i) $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & -4 & 4 \\ 1 & -2 & 4 \\ 2 & -1 & 3 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ (v) $\begin{bmatrix} 2 & -3 \\ 4 & -2 \end{bmatrix}$

Soln :

(i) Sum of the eigen values = Sum of the main diagonal elements

$= -2 + 1 + 0$

$= -1$

Product of the eigen values = $|A|$

$$= \begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix}$$

$$= -2(-12) - 2(0-6) - 3(-4+1)$$

$$= 24 + 12 + 9 = 45$$

(ii) Sum $= (-1) + (-1) + (-1) = -3$

Product $= \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$

$$= -1(1-1) - 1(-1-1) + 1(1+1)$$

$$= -1(0) - 1(-2) + 1(2)$$

$$= 4$$

(iii) Sum = 0

Product = -10

(iv) Sum = 4

Product = -4

(v) Sum = 0

Product = 8

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(2) Find the constants a and b such that the matrix $\begin{pmatrix} a & 4 \\ 1 & b \end{pmatrix}$ has 3 and -2 as its eigen values.

Soln: Sum of the eigen values = Sum of the main diagonal elements

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$3 + (-2) = a + b$

$a + b = 1 \longrightarrow (1)$

Product of the eigen values $= |A|$

$(3)(-2) = \begin{vmatrix} a & 4 \\ 1 & b \end{vmatrix}$

$-6 = ab - 4$

$ab = -2 \longrightarrow (2)$

$(a-b)^2 = (a+b)^2 - 4ab$ $\quad$ $\cancel{(a+b)^2 = a^2 + b^2 + 2ab}$

$= 1 - 4(-2)$ $\quad$ $a - \frac{2}{a} = 1$

$(a-b)^2 = 9$

$\therefore a - b = \pm 3 \longrightarrow (3)$ $\quad$ $ab = -2$ $\quad$ $a^2 - 2 = a$

Taking $a - b = 3$ $\quad$ $b = \frac{-2}{a}$ $\quad$ $a^2 - a = 2$ $\quad$ $a(a-1) = 2$

$a = 2 \Rightarrow b = -1$ $\quad$ $a^2 - a - 2 = 0$

$(1) + (3) \Rightarrow a + b = 1$ $\quad$ $a = -1,\ b = 2$

$a - b = 3$

$2a = 4 \Rightarrow a = 2$ $\quad$ $\boxed{a = 2, -1}$

put $a = 2$ in $a + b = 1 \Rightarrow b = 1 - 2$

$\boxed{b = -1}$

$\boxed{a = 2,\ b = -1}$

Taking $a - b = -3$, we get

$\boxed{a = -1,\ b = 2}$

(3) (i) Two of the eigen values of $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ are 2 & 8. Find the third eigen value. What is the Product of the eigen values.

(ii) Two eigen values of $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ are 3 & 0. Find the third eigen value? What is the Product of the eigen values of A.

(i) Sum of the eigen values

= Sum of the main diagonal elts

$= 6+3+3$

$= 12.$

Given : $\lambda_1 = 2, \lambda_2 = 8, \lambda_3 = ?$

$\lambda_1 + \lambda_2 + \lambda_3 = 12$

$2 + 8 + \lambda_3 = 12$

$\boxed{\lambda_3 = 2}$

Product of the eigen value $= \lambda_1 \lambda_2 \lambda_3$

$= 2 \times 8 \times 2$

$= 32.$

(ii) $\lambda_3 = 15$

Product = 0.

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(4) The product of the two eigen values of the matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ is 16. Find the third eigen value.

Soln:

Given : $\lambda_1 \lambda_2 = 16$

Product of eigen values $= |A|$

$$\lambda_1 \lambda_2 \lambda_3 = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$

$$= 6(9-1) + 2(-6+2) + 2(2-6)$$

$\lambda_1 \lambda_2 \lambda_3 = 32$

Case (ii) : $\lambda = -3$

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

All the three equations are same.

$$x_1 + 2x_2 - 3x_3 = 0$$

Put $x_1 = 0$,

$$2x_2 = 3x_3$$

$$\frac{x_2}{3} = \frac{x_3}{2}$$

$$\therefore \quad X_2 = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$$

Put $x_2 = 0$,

$$x_1 = 3x_3$$

$$\frac{x_1}{3} = \frac{x_3}{1}$$

$$\therefore \quad X_3 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$$

Eigen values : $5 \quad -3 \quad -3$

Eigen vectors : $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$

(2) $\begin{pmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{pmatrix}$ Soln : $\lambda : -2 \quad 2 \quad 2$

$X : \begin{pmatrix} 4 \\ 1 \\ -7 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

(3) $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ Soln : $\lambda : 1 \quad 1 \quad 5$

$X : \begin{pmatrix} 2 \\ -1 \\ \cdot \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \cdot \end{pmatrix}$

$16\lambda_3 = 32$

$\boxed{\lambda_3 = 2}$

(5) One of the eigen value of $\begin{bmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ 4 & -1 & -8 \end{bmatrix}$ is -9.

Find the other two eigen values

Soln:

Let $A = \begin{bmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ 4 & -1 & -8 \end{bmatrix}$

Sum of the eigen values = Sum of the main diagonal elements.

$\lambda_1 + \lambda_2 + \lambda_3 = 7 - 8 - 8 = -9$

But $\lambda_1 = -9$ (given).

$\therefore \lambda_2 + \lambda_3 - 9 = -9$

$\therefore \lambda_2 + \lambda_3 = 0 \longrightarrow (1)$

Product of the eigen values $= |A|$.

$\lambda_1\lambda_2\lambda_3 = \begin{vmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ 4 & -1 & -8 \end{vmatrix}$

$-9\lambda_2\lambda_3 = 7(64-1) - 4(-32+4) - 4(-4+32)$

$= 441 + 112 - 112 = 441$

$\lambda_2\lambda_3 = \frac{-441}{9} = -49$

$\lambda_2\lambda_3 = -49 \longrightarrow (2)$

From (1), $\lambda_2 = -\lambda_3$

Subs in (3),

$$-\lambda_3^2 = -49$$

$$\lambda_3^2 = 49 \Rightarrow \lambda_3 = \pm 7$$

If $\lambda_3 = 7$, $\lambda_2 = -7$

If $\lambda_3 = -7$, $\lambda_2 = 7$

Hence the two eigen values are 7 and -7

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6 (i) If 2, 2, 3 are the eigen values of $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$ find the eigen values of A^T

(ii) If the eigen values of the matrix $A = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$ are $2, -2$ then find the eigen values of A^T.

Soln:

(i) By property 7, eigen values of the matrices A & A^T are same.

A^T has eigen values 2, 2 & 3

(ii) $2, -2$.

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7 (i) Find the eigen values of A^2, given $A = \begin{pmatrix} -1 & 0 & 0 \\ 2 & -3 & 0 \\ 1 & 4 & 2 \end{pmatrix}$

(ii) Find the eigen values of A^3 given $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & -7 \\ 0 & 0 & 3 \end{pmatrix}$

Soln:

(i) The eigen values of A are $-1, -3, 2$

The eigen values of A^2 are $(-1)^2, (-3)^2, 2^2$

i.e., $1, 9, 4$

(ii) The eigen values of A are $1, 2, 3$

The eigen values of A^3 are $1^3, 2^3, 3^3$

i.e., $1, 8, 27$.

(8) If -1 is an eigen value of the matrix $A = \begin{pmatrix} 1 & -2 \\ 3 & 2 \end{pmatrix}$, then find the eigen values of A^4 using properties.

Soln: Given: $A = \begin{pmatrix} 1 & -2 \\ 3 & 2 \end{pmatrix}$

$\lambda_1 = -1, \ \lambda_2 = ?$

Sum of the eigen values = Sum of the main diagonal elements

$\lambda_1 + \lambda_2 = 1 + 2 = 3$

$-1 + \lambda_2 = 3 \Rightarrow \lambda_2 = 4$

$\lambda_1 = -1, \ \lambda_2 = 4$

The eigen values of A^4 are λ_1^4, λ_2^4

i.e., $(-1)^4, 4^4 \Rightarrow 1, 256$ are the eigen values of A^4.

(9) (i) Two eigen values of $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ are equal and they are $\frac{1}{5}$ times to the third. Find them.

(ii) Two eigen values of $A = \begin{pmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{pmatrix}$ are equal and they are double the third. Find the eigen values of A^2.

Soln:

(i) Let the third eigen value be λ_3.

Given: $\lambda_1 = \frac{1}{5}\lambda_3$, $\lambda_2 = \frac{1}{5}\lambda_3$.

Sum of the eigen values = Sum of the main diagonal elements.

$$\frac{1}{5}\lambda_3 + \frac{1}{5}\lambda_3 + \lambda_3 = 2 + 3 + 2$$

$$\left(\frac{1}{5} + \frac{1}{5} + 1\right)\lambda_3 = 7$$

$$\frac{7}{5}\lambda_3 = 7 \Rightarrow \boxed{\lambda_3 = 5}$$

$$\lambda_1 = \frac{1}{5} \cdot 5 = 1 \Rightarrow \boxed{\lambda_1 = 1}$$

$$\lambda_2 = \frac{1}{5} \cdot 5 = 1 \Rightarrow \boxed{\lambda_2 = 1}$$

(ii) Let the third eigen value be λ_3.

Given: $\lambda_1 = 2\lambda_3$, $\lambda_2 = 2\lambda_3$

Sum of the eigen values = Sum of the main diagonal elements

$$2\lambda_3 + 2\lambda_3 + \lambda_3 = 4 + 3 - 2$$

$$5\lambda_3 = 5$$

$$\boxed{\lambda_3 = 1}$$

$$\boxed{\begin{array}{l}\lambda_1 = 2 \\ \lambda_2 = 2\end{array}}$$

The eigen values of A^2 are $2^2, 2^2, 1^2$.
i.e, 4, 4, 1.

(10) If 1, 1, 5 are the eigen values of $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ find the eigen values of 5A.

Soln:

If $\lambda_1, \lambda_2, \lambda_3$ be the eigen values of A
Then $k\lambda_1, k\lambda_2, k\lambda_3$ are the eigen values of KA
The eigen values of 5A are 5, 5, 25.

(11) Find the eigen values of the matrix A^{-1} if $A = \begin{pmatrix} 1 & 5 \\ 0 & 4 \end{pmatrix}$.

Soln:

By property, "If λ be an eigen value of a non-singular matrix A, then λ^{-1} is an eigen value of A^{-1} and $\frac{|A|}{\lambda}$ is an eigen value of adj A".

Given: $A = \begin{pmatrix} 1 & 5 \\ 0 & 4 \end{pmatrix}$

$|A| = 4 \neq 0$.

$\therefore$ The eigen value of A^{-1} are $\frac{1}{1}, \frac{1}{4}$.

(12) Find the eigen value of A^{-1} if the matrix
$A = \begin{pmatrix} 2 & 5 & -1 \\ 0 & 3 & 2 \\ 0 & 0 & 4 \end{pmatrix}$

Soln:

A has eigen values 2, 3, 4

$\therefore$ By property, A^{-1} has eigen values $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$

(13) Find the eigen values of adj A, if

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Soln:

A is an upper triangular matrix.

The eigen values of A are 3, 4, 1.

We know that,

$$A^{-1} = \frac{1}{|A|} \text{adj } A.$$

$$\text{adj } A = |A| \cdot A^{-1}$$

Eigen values of adj A = |A|. eigen values of A^{-1} $\longrightarrow$ (1)

$$|A| = \begin{vmatrix} 3 & 2 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 3(4) = 12.$$

Eigen value of $A^{-1} = \frac{1}{3}, \frac{1}{4}, 1$

$\therefore$ (1) $\Rightarrow 12 \cdot \frac{1}{3}, \frac{1}{4}, 1$

$\Rightarrow \frac{12}{3}, \frac{12}{4}, 12$

$= 4, 3, 12$ are the eigen values of adj A.

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$$\boxed{\begin{array}{l}\lambda_1 = 2 \\ \lambda_2 = 2\end{array}}$$

The eigen values of A^2 are $2^2, 2^2, 1^2$.

i.e., 4, 4, 1.

(10) If 1, 1, 5 are the eigen values of $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ find the eigen values of 5A.

Soln:

If $\lambda_1, \lambda_2, \lambda_3$ be the eigen values of A

Then $k\lambda_1, k\lambda_2, k\lambda_3$ are the eigen values of KA

The eigen values of 5A are 5, 5, 25.

(11) Find the eigen values of the matrix A^{-1} if $A = \begin{pmatrix} 1 & 5 \\ 0 & 4 \end{pmatrix}$.

Soln:

By property, "If λ be an eigen value of a non-singular matrix A, then λ^{-1} is an eigen value of A^{-1} and $\frac{|A|}{\lambda}$ is an eigen value of adj A".

Given: $A = \begin{pmatrix} 1 & 5 \\ 0 & 4 \end{pmatrix}$

$|A| = 4 \neq 0$.

$\therefore$ The eigen value of A^{-1} are $\frac{1}{1}, \frac{1}{4}$.

(12) Find the eigen value of A^{-1} if the matrix

$A = \begin{pmatrix} 2 & 5 & -1 \\ 0 & 3 & 2 \\ 0 & 0 & 4 \end{pmatrix}$

(14) If the eigenvalues of A are $1, 2, 3$ then what are the eigen values of adj A ?

Soln:

Eigen values of A are $1, 2, 3$

Eigen values of A^{-1} are $1, \frac{1}{2}, \frac{1}{3}$.

We know that,

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$\text{adj } A = |A| \, A^{-1}$$

$|A|$ = product of eigen values

$$= (1)(2)(3)$$
$$= 6.$$

$\therefore$ adj $A = 6(1), 6\left(\frac{1}{2}\right), 6\left(\frac{1}{3}\right) = 6, 3, 2$

(15) If 3 and 6 are two eigen values of $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$, write down all the eigen values of A^{-1}.

Soln:

Given: $\lambda_1 = 3$, $\lambda_2 = 6$.

Sum of the eigen value = Sum of the main diagonal elements.

$$\lambda_1 + \lambda_2 + \lambda_3 = 1 + 5 + 1$$

$$3 + 6 + \lambda_3 = 7 \Rightarrow \boxed{\lambda_3 = -2}$$

$\therefore$ The eigen values of A^{-1} are $\frac{1}{3}, \frac{1}{6}, \frac{-1}{2}$

(16) Prove that eigen values of $-3A^{-1}$ are the same as those of $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

Soln:

The char eqn is,

$\lambda^2 - c_1\lambda + c_2 = 0.$

$c_1 = 1+1 = 2$

$c_2 = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 1-4 = -3$

$\lambda^2 - 2\lambda - 3 = 0.$

$\lambda = 3, -1$

The eigen values of A are $-1, 3$.

The eigen values of $-3A^{-1}$ are $(-3)\frac{1}{(-1)}, \frac{(-3)}{3}$

i.e., $3, -1$

Hence the proof.

(17) If the sum of two eigen values and the trace of a 3×3 matrix A are equal, find the value of $|A|$.

Soln:

Let $\lambda_1, \lambda_2, \lambda_3$ be the eigen values of A.

Given: $\lambda_1 + \lambda_2 =$ Sum of the main diagonal elements

$= \lambda_1 + \lambda_2 + \lambda_3$

$\Rightarrow \lambda_3 = 0.$

$\therefore |A| =$ product of the eigen values

$= \lambda_1\lambda_2\lambda_3$

$= 0$

(18) If the eigenvalues of $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ are $-1, -1, 2$ and if two of the eigen vectors of A are $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, then find the third eigen vector.

Soln:

Given A is a symmetric matrix.

Let $X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $X_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

X_3 is orthogonal to X_1 & X_2.

Let $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

X_3 is orthogonal to $X_1 \Rightarrow X_3^T X_1 = 0$

$$(a \;\; b \;\; c) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0$$

$$a + b + c = 0 \rightarrow (1)$$

X_3 is orthogonal to $X_2 \Rightarrow X_3^T X_2 = 0$

$$(a \;\; b \;\; c) \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = 0$$

$$0a + b - c = 0 \rightarrow (2)$$

Solving (1) & (2)

$$\begin{matrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 \end{matrix}$$

$$\frac{x_1}{-2} = \frac{x_2}{1} = \frac{x_3}{1} \qquad \therefore X_3 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

(19) If X_1, X_2, X_3 are the eigen vectors of a matrix A, what are the eigen vectors of A^2

Soln:

Eigen vectors of A and A^2 are same

$\therefore X_1, X_2, X_3$ are the eigen vectors of A^2.

(20) One of the eigen value of a matrix is zero. What do you know about the matrix?

Soln:

If one of the eigen value is zero then the product of the eigen value is zero.

$|A|$ = product of eigen value $= 0$

$\therefore$ The matrix is singular.