SNS COLLEGE OF TECHNOLOGY

(An Autonomous Institution)

Coimbatore-641035.

UNIT I- MATRIX EIGENVALUE PROBLEMS   Eigen values and Eigen vectors of a real matrix

1. Find the Eigen value & Eigen vectors of $A = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$

Step 1: Characteristic eqn : $\lambda^2 - C_1\lambda + C_2 = 0$.

C_1 = Sum of main diagonal elements $= 1 - 1 = 0$

$C_2 = |A| = -1 - 3 = -4$.

$\therefore \lambda^2 - 4 = 0$

Step 2: To find Eigen Values.

$\lambda^2 = 4 \Rightarrow \lambda = \pm 2$.

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$\therefore$ The Eigen values are 2 and -2.

Step 3: To find the Eigen vectors

$(A - \lambda I) X = 0$

$$\left[\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Case 1: When $\lambda = -2$

$$\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$3x_1 + x_2 = 0. \Rightarrow 3x_1 = -x_2 \Rightarrow \frac{x_1}{-1} = \frac{x_2}{3}$

Eigen vectors of $X_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

Case 2: when $\lambda = 2$.

$$\begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$-x_1 + x_2 = 0. \quad - (3)$

$3x_1 - 3x_2 = 0. \quad - (4)$

$\because$ (3) & (4) are the same take

$-x_1 + x_2 = 0. \Rightarrow x_2 = x_1 \Rightarrow \frac{x_1}{1} = \frac{x_2}{1}$

$\therefore$ Eigen vectors are $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

2. Find the E.V & E.Vectors of $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

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UNIT I- MATRIX EIGENVALUE PROBLEMS — Eigen values and Eigen vectors of a real matrix

Step 1: Characteristic eq.

$$\lambda^3 - C_1\lambda^2 + C_2\lambda - C_3 = 0.$$

$C_1 = 8+7+3 = 18$

$C_2 =$ Sum of minors of main diagonal el.

$$= \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}$$

$= 5 + 20 + 20 = 45$

$$C_3 = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix} = 8(5) + 6(-10) + 2(10)$$

$= 40 - 60 + 20 = 0.$

45

$-15 \quad -3$

$\therefore$ The C.E is $\lambda^3 - 18\lambda^2 + 45\lambda = 0.$

Step 2: E.V $\quad \lambda[\lambda^2 - 18\lambda + 45] = 0.$

$\lambda = 0, \ \lambda = 3, 15.$

$\therefore$ The Eigen values are $0, 3, 15$

Step 3: E.vectors. $\quad (A - \lambda I)X = 0.$

$$\left[\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\begin{pmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Case i) when $\lambda = 0$

$8x_1 - 6x_2 + 2x_3 = 0$ —(1)

$-6x_1 + 7x_2 - 4x_3 = 0$ —(2)

$2x_1 - 4x_2 + 3x_3 = 0$ —(3)

x_1		x_2		x_3
-6	2	8	-6	
7	-4	-6	7	

$$\frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20}$$

Considering (1) & (2) by omitting (3) we get

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$$\frac{x_1}{\begin{vmatrix} -6 & 2 \\ 7 & -4 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 8 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}}$$

$$\Rightarrow \frac{x_1}{10} = \frac{-x_2}{-20} = \frac{x_3}{20} \Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$$

$\therefore$ Eigen vectors is $X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

Case 2: When $\lambda = 3$.

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$5x_1 - 6x_2 + 2x_3 = 0$; $-6x_1 + 4x_2 - 4x_3 = 0$; $2x_1 - 4x_2 + 0x_3 = 0$

$$\frac{x_1}{\begin{vmatrix} -6 & 2 \\ 4 & -4 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 5 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 5 & -6 \\ -6 & 4 \end{vmatrix}}$$

$$\frac{x_1}{16} = \frac{-x_2}{-8} = \frac{x_3}{-16} \Rightarrow \frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$$

$\therefore X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

Case 3: When $\lambda = 15$

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-7x_1 - 6x_2 + 2x_3 = 0$; $-6x_1 - 8x_2 - 4x_3 = 0$; $2x_1 - 4x_2 - 12x_3 = 0$

$$\frac{x_1}{\begin{vmatrix} -6 & 2 \\ -8 & -4 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -7 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -7 & -6 \\ -6 & -8 \end{vmatrix}}$$

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$$\frac{x_1}{40} = \frac{-x_2}{40} = \frac{x_3}{20} \Rightarrow \frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$$

$$X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

Conclusion:

Characteristic eq.	E.V	Eigen Vectors
$\lambda^3 - 18\lambda^2 + 45\lambda = 0$	$\lambda = 0$	$\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$
	$\lambda = 3$	$\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$
	$\lambda = 15$	$\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

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Repeated E.V

5. Find all the E.V & E vectors of the matrix.

$$A = \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$$

Step 1: $\lambda^3 - c_1\lambda^2 + c_2\lambda - c_3 = 0$

$c_1 = -1$, $c_2 = -12 - 3 - 6 = -21$, $c_3 = 45$

$\therefore$ C.E is $\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$.

Step 2:

$$\begin{array}{r|rrrr} -3 & 1 & 1 & -21 & -45 \\ & 0 & -3 & 6 & 45 \\ \hline & 1 & -2 & -15 & \boxed{0} \end{array}$$

-15 ; -5, 3

$\therefore (\lambda+3)(\lambda^2 - 2\lambda - 15) = 0$.

$(\lambda+3)(\lambda-5)(\lambda+3) = 0 \Rightarrow \lambda = 5, -3, -3$.

Step 3: $(A - \lambda I)x = 0$.

$$\left[\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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Case i) If $\lambda = -3$, then.

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + 2x_2 - 3x_3 = 0$; $2x_1 + 4x_2 - 6x_3 = 0$; $-x_1 - 2x_2 + 3x_3 = 0$

all the eqs are same.

$\therefore x_1 + 2x_2 - 3x_3 = 0.$

Put $x_1 = 0 \Rightarrow 2x_2 = 3x_3 \Rightarrow \frac{x_2}{3} = \frac{x_3}{2}.$

$\therefore X_1 = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$

Put $x_2 = 0$., we get $x_1 - 3x_3 = 0 \Rightarrow \frac{x_1}{3} = \frac{x_3}{1}$

$\Rightarrow X_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$

$\because$ The matrix is non-symmetric the corresponding Eigen Vectors X_1 & X_2 must be Linearly Independent

Case ii) If $\lambda = 5$, then.

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-7x_1 + 2x_2 - 3x_3 = 0$; $2x_1 - 4x_2 - 6x_3 = 0$; $-x_1 - 2x_2 - 5x_3 = 0$

$$\frac{x_1}{\begin{vmatrix} 2 & -3 \\ -4 & -6 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -7 & -3 \\ 2 & -6 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -7 & 2 \\ 2 & -4 \end{vmatrix}}$$

$$\frac{x_1}{-24} = \frac{-x_2}{48} = \frac{x_3}{24}$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{-1} \quad \therefore \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

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6) $A = \begin{pmatrix} 6 & -6 & 5 \\ 14 & -13 & 10 \\ 7 & -6 & 4 \end{pmatrix}$ $\quad \begin{pmatrix} 0 \\ 5 \\ 6 \end{pmatrix} \begin{pmatrix} -5 \\ 0 \\ 7 \end{pmatrix} \begin{pmatrix} 6 \\ 7 \\ 0 \end{pmatrix}$

Step 1: $\lambda^3 - c_1\lambda^2 + c_2\lambda - c_3 = 0$.

$c_1 = 6 - 13 + 4 = -3$.

$c_2 = \begin{vmatrix} -13 & 10 \\ -6 & 4 \end{vmatrix} + \begin{vmatrix} 6 & 5 \\ 7 & 4 \end{vmatrix} + \begin{vmatrix} 6 & -6 \\ 14 & -13 \end{vmatrix}$

$= 8 - 11 + 6 = 3$.

$c_3 = \begin{vmatrix} 6 & -6 & 5 \\ 14 & -13 & 10 \\ 7 & -6 & 4 \end{vmatrix} = 6[8] + 6[-14] + 5[7] = 48 - 84 + 35 = -1$

$\therefore$ C.E is $\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$.

Step 2: $\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$.

$$\begin{array}{r|rrrr} -1 & 1 & 3 & 3 & 1 \\ & 0 & -1 & -2 & -1 \\ \hline & 1 & 2 & 1 & \boxed{0} \end{array}$$

$\Rightarrow (\lambda+1)(\lambda^2+2\lambda+1) = 0$.

$\Rightarrow (\lambda+1)(\lambda+1)(\lambda+1) = 0$.

$\therefore$ E.values are $-1, -1, -1$.

Step 3: Eigen vector $(A - \lambda I)X = 0$.

$$\left[\begin{pmatrix} 6 & -6 & 5 \\ 14 & -13 & 10 \\ 7 & -6 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{pmatrix} 6-\lambda & -6 & 5 \\ 14 & -13-\lambda & 10 \\ 7 & -6 & 4-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

When $\lambda = -1$ we get $\begin{pmatrix} 7 & -6 & 5 \\ 14 & -12 & 10 \\ 7 & -6 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

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$7x_1 - 6x_2 + 5x_3 = 0$; $14x_1 - 12x_2 + 10x_3 = 0$; $7x_1 - 6x_2 + 5x_3 = 0$

All the eqs are same.

$\therefore$ Put $x_1 = 0 \Rightarrow -6x_2 + 5x_3 = 0 \Rightarrow \frac{x_2}{5} = \frac{x_3}{6}$

$\therefore \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 6 \end{pmatrix}$ Put $x_2 = 0 \Rightarrow 7x_1 + 5x_3 = 0 \Rightarrow \frac{x_1}{-5} = \frac{x_3}{7}$

$\therefore \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -5 \\ 0 \\ 7 \end{pmatrix}$ Put $x_3 = 0 \Rightarrow 7x_1 - 6x_2 = 0 \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \\ 0 \end{pmatrix}$

$\because$ The matrix is non-symmetric, the corresponding E-vectors X_1, X_2 & X_3 must be L.I for diagonalisation