UNIT III
DIFFERENTIAL CALCULUS

(17)

Problems:

① Find the radius of curvature at any point of the catenary $y = c\cosh(x/c)$

Soln:

$$y = c\cosh(x/c) \longrightarrow ①$$

$$y_1 = c \cdot \sinh(x/c) \cdot \frac{1}{c} = \sinh(x/c)$$

$$y_2 = \cosh\left(\frac{x}{c}\right) \cdot \frac{1}{c}$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$\cosh^2 x - \sinh^2 x = 1$

$$= \frac{\left(1+\sinh^2(x/c)\right)^{3/2}}{\cosh(x/c)} \cdot c$$

$$= \frac{\left(\cosh^2(x/c)\right)^{3/2}}{\cosh(x/c)} \cdot c$$

$$\rho = c\cosh^2(x/c) \longrightarrow ②$$

From ①, $\cosh(x/c) = y/c$

subs this in ②, we get,

$$\rho = c \cdot y^2/c^2$$

$$\boxed{\rho = y^2/c}$$

(2) Find ρ for (i) $y = \log \sin x$ (ii) $y = c \log \sec(x/c)$

Soln:

(i) $y = \log \sin x$

$$y_1 = \frac{1}{\sin x} \cos x = \cot x$$

$$y_2 = -\operatorname{cosec}^2 x$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+\cot^2 x)^{3/2}}{-\operatorname{cosec}^2 x}$$

$$= \frac{\operatorname{cosec}^3 x}{-\operatorname{cosec}^2 x} = -\operatorname{cosec} x$$

$$\boxed{|\rho| = \operatorname{cosec} x}$$

(ii) $y = c \log \sec(x/c)$

$$y_1 = c \frac{1}{\sec(x/c)} \sec\left(\frac{x}{c}\right) \tan\left(\frac{x}{c}\right) \cdot \frac{1}{c}$$

$$y_1 = \tan(x/c)$$

$$y_2 = \frac{1}{c} \sec^2\left(\frac{x}{c}\right)$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{\left(1+\tan^2(x/c)\right)^{3/2}}{\frac{1}{c}\sec^2(x/c)}$$

$$\rho = c \cdot \frac{\sec^3(x/c)}{\sec^2(x/c)} \Rightarrow \boxed{\rho = c \sec(x/c)}$$

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(3) Find the radius of curvature for the curves

(i) $y = e^x$, where it crosses the y-axis

(ii) $y^2 = x^3 + 8$ at the point $(-2, 0)$

(iii) $xy = c^2$ at $x = c$

(iv) $y^2 = \frac{a^3 - x^3}{x}$ at $(a, 0)$

(v) $y = \frac{\log x}{x}$ at $x = 1$

(vi) $x^3 + y^3 = 3axy$ at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

Soln:

(i) $y = e^x$

On y-axis $x = 0$

when $x = 0$, $y = e^0 = 1$ $\therefore y = 1$

$\therefore$ The point is $(0, 1)$

$y = e^x$

$y_1 = e^x$; y_1 at $(0,1) = 1$

$y_2 = e^x$; y_2 at $(0,1) = 1$

$$\therefore \rho = \frac{(1 + y_1^2)^{3/2}}{y_2} = \frac{(1+1)^{3/2}}{1} = 2^{3/2}$$

$$\boxed{\rho = 2\sqrt{2}}$$

(ii) $y^2 = x^3 + 8$

Diff w.r.t 'x', we get

$2yy_1 = 3x^2$

$y_1 = \frac{3x^2}{2y}$; $y_1(-2,0) = \infty$

$\therefore \frac{dx}{dy} = 0$; $\frac{dx}{dy} = \frac{2y}{3x^2}$

$$\frac{d^2x}{dy^2} = \frac{(3x^2)2 - (2y)6x\, dx/dy}{(9x^4)}$$

$$= \frac{6x^2 - 12xy\,(dx/dy)}{9x^4}$$

$$\left(\frac{d^2x}{dy^2}\right)_{(-2,0)} = \frac{6\times 4}{9\times 16} = \frac{1}{6}$$

$$\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{d^2x/dy^2} = \frac{(1+0)^{3/2}}{1/6}$$

$\boxed{\rho = 6}$

(iii) $xy = c^2$ at $x = c$

If $x = c$, $xy = c^2$ will be

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$y = c^2/x = c^2/c = c$

$y = c$

$\therefore$ The point is (c, c).

$xy = c^2$

$x y_1 + y = 0$

$y_1 = -y/x$

y_1 at (c,c) is $= -c/c = -1$

$$y_2 = \frac{x(-dy/dx) - (-y)\cdot 1}{x^2}$$

$$= \frac{-x \times (-y/x) + y}{x^2} = \frac{2y}{x^2}$$

y_2 at $(c,c) = \frac{2c}{c^2} = \frac{2}{c}$

$$\therefore \rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$= \frac{(1+(-1)^2)^{3/2}}{2/c} = \frac{2^{3/2} c}{2}$$

$$= \frac{2\sqrt{2}\, c}{2}$$

$$\boxed{\rho = \sqrt{2} \cdot c}$$