Properties of Group:

1]. The identity element in a group is unique.

2]. The inverse element in a group is unique

3]. Cancellation law:

$\forall a, b \in G$ i). $a * b = a * c \Rightarrow b = c$ [Left cancellation]

ii). $b * a = c * a$ [Right cancellation]

$\Rightarrow b = c$

4). Let G be a group.

If $a, b \in G$ then $(a * b)^{-1} = b^{-1} * a^{-1}$

Proof:

Let $a, b \in G$

and a^{-1}, b^{-1} be their inverses resly.

Let $(G, *)$ be a group.

Let $a, b \in G$. $a * a^{-1} = a^{-1} * a = e$

$b * b^{-1} = b^{-1} * b = e$

Now $(a * b) * (b^{-1} * a^{-1}) = a * (b * b^{-1}) * a^{-1}$ (Associative)

$= (a * e) * a^{-1}$

$= a * a^{-1}$ Identity

$(a * b) * (b^{-1} * a^{-1}) = e \rightarrow (1)$

Ily $(b^{-1} * a^{-1}) * (a * b) = e \rightarrow (2)$

From (1) and (2), $(a * b)^{-1} = b^{-1} * a^{-1}$

The inverse of $(a * b)$ is $b^{-1} * a^{-1}$

Hence proved.

5]. Prove that a group $(G, *)$ is abelian iff

$(a * b)^2 = a^2 * b^2, \forall a, b \in G$

Proof:

Assume that G is abelian

$\therefore a * b = b * a, \quad a, b \in G \rightarrow (1)$

Now $a^2 * b^2 = (a*a) * (b*b)$

$= a*[a*(b*b)]$ Associative

$= a*[(a*b)*b]$ Associative

$= a*[(b*a)*b]$ Com By (1)

$= (a*b)*(a*b)$ Associative

$= (a*b)^2$

conversely,
Assume that $(a*b)^2 = a^2 * b^2$

$(a*b)*(a*b) = (a*a)*(b*b)$

$a*[b*(a*b)] = a*[a*(b*b)]$

$b*(a*b) = a*(b*b)$ left cancellation law

$(b*a)*b = (a*b)*b$

$b*a = a*b$ Right cancellation law

$\Rightarrow G$ is abelian.

6]. If $(G,*)$ is an abelian group, ST $(a*b)^n = a^n * b^n$, $\forall a, b \in G$, where n is a +ve integer

proof:

Since $(G,*)$ is abelian, we've

$a*b = b*a, \quad \forall a, b \in G \longrightarrow (1)$

For $a, b \in G$, we've $(a*b)^1 = (b*a)^1$ by (1)

and $(a*b)^2 = (a*b)*(a*b)$

$= a*(b*a)*b$ Associative

$= a*(a*b)*b$ by (1)

$= (a*a)*(b*b)$ Associative

$= a^2 * b^2$

Thus, the result is true for $n = 1, 2, \ldots$

Let us assume that the result is valid for $n = m$.

$$(a*b)^m = a^m * b^m \quad \rightarrow (2)$$

Now $(a*b)^{m+1} = (a*b)^m * (a*b)$

$= (a^m * b^m) * (a*b)$ By (2)

$= a^m * (b^m * a) * b$ Associative

$= a^m * (a * b^m) * b$ (G is abelian)

$= (a^m * a) * (b^m * b)$

$(a*b)^{m+1} = a^{m+1} * b^{m+1}$

Hence by induction, the result is true for +ve integral values of n.

7). In a group G, Prove that an elt. $a \in G$ such that $a^2 = e$, $a \neq e$ iff $a = a^{-1}$

Proof:

Assume that $a = a^{-1}$

To prove: $a^2 = e$

Now consider $a^2 = a * a$

$= a * a^{-1}$

$= e$

Conversely, Assume $a^2 = e$

To prove: $a = a^{-1}$

Now $a^2 = e$

$a * a = e$

$a^{-1} * (a*a) = a^{-1} * e$ [Premultiply by a^{-1}]

$(a^{-1} * a) * a = a^{-1}$

$e * a = a^{-1}$

$a = a^{-1}$.

8). If ery. elt. of a group G has its own inverse, then G is abelian.

Proof:

Let $(G, *)$ be a group.

for $a, b \in G$, we've $a * b \in G$

Given $a = a^{-1}$ and $b = b^{-1}$

Now $a * b = (a * b)^{-1}$ [It has its own Inverse]

$= b^{-1} * a^{-1}$ (By property)

$= b * a$

$\therefore$ G is abelian

Converse need not be true.