Eigen Value Problems

Characteristic Equations

If A is a square matrix of order n, we can form the matrix $A-\lambda I$, where λ is a scalar and I is the unit matrix of order n, then $|A-\lambda I| = 0$ is called the characteristic equation.

The determinant $|A-\lambda I|$ when expanded will give a polynomial, which is called as a characteristic polynomial of matrix A.

Note:

1) For any square matrix A, the sum of the eigen values of a matrix is equal to trace of the matrix.

2) For a 2×2 matrix, the characteristic equ is, $\lambda^2 - c_1\lambda + c_2 = 0$,
where c_1 = sum of the main diagonal elements
$c_2 = |A|$.

3) For a 3×3 matrix, the characteristic equ is,
$\lambda^3 - c_1\lambda^2 + c_2\lambda - c_3 = 0$
where c_1 = Sum of the main diagonal elements
c_2 = Sum of the minors of main diagonal elements.
$c_3 = |A|$.

Problems:

1) Find the characteristic equation of $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$

Soln:

Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$

The char equ is, $\lambda^2 - c_1\lambda + c_2 = 0$

c_1 = sum of the diagonal elements

$c_1 = 1+2 = 3.$

$c_2 = |A| = \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} = 2 - 0 = 2.$

$\therefore \lambda^2 - 3\lambda + 2 = 0$ is the char equ.

2) Find the char equ of $\begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix}$.

Soln:

Let $A = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix}$

The char equ is, $\lambda^3 - c_1\lambda^2 + c_2\lambda - c_3 = 0.$

$c_1 = 2 + 1 - 4 = -1.$

$c_2 = \begin{vmatrix} 1 & 3 \\ 2 & -4 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ -5 & -4 \end{vmatrix} + \begin{vmatrix} 2 & -3 \\ 3 & 1 \end{vmatrix}$

$= (-4-6) + (-8+5) + (2+9) = -10 - 3 + 11 = -2.$

$c_3 = \begin{vmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{vmatrix} = 2(-4-6) - (-3)(-12+15) + 1(6+5)$
$= -20 + 9 + 11 = 0$

$\therefore$ The char equ is, $\lambda^3 + \lambda^2 - 2\lambda = 0.$

3) If the char equ of $\begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 1 \\ -7 & 2 & 3 \end{pmatrix}$ is $\lambda^3 + a\lambda^2 + b\lambda + 18 = 0$, find the values of a & b.

Soln:

a = sum of the diagonal elements

$= 2+2+3$

$\boxed{a = 7}$

b = Sum of the minors of the diagonal elements

$$= \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ -7 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 2 & 2 \end{vmatrix}$$

$$= 6-2+6-0+4-4$$

$$\boxed{b = 10}$$

4) Write the two matrices with $\lambda^2 - 7\lambda + 6 = 0$ as the char equ.

Soln: Since $\lambda^2 - 7\lambda + 6 = 0$ is the char equ.

$\therefore c_1 = 7, \quad c_2 = 6$

The two matrices are

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix} \quad \& \quad B = \begin{pmatrix} 6 & 0 \\ 4 & 1 \end{pmatrix}$$

5) Find the char polynomial of

(i) $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ (Soln: $\lambda^2 - 4\lambda - 5$)

(ii) $\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$ (Soln: $\lambda^2 - 5\lambda + 7$)

6) Find the char equ of

(i) $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ (Soln: $\lambda^3 - 18\lambda^2 + 45\lambda = 0$)

(ii) $\begin{bmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{bmatrix}$ (Soln: $\lambda^3 - 3\lambda^2 + 2\lambda = 0$)

(iii) $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ (Soln: $\lambda^3 - 7\lambda^2 + 36 = 0$).