UNIT II
ORTHOGONAL TRANSFORMATION OF A REAL SYMMETRIC MATRIX

Transformation to principal axes :

Conic section :

(1) Find out what type of conic section the following quadratic form represents :

(i) $Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$

(ii) $Q = 3x_1^2 + 22x_1x_2 + 3x_2^2 = 0$

(iii) $Q = x_1^2 - 12x_1x_2 + x_2^2 = 70$.

Solution :

(i) $Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = 128 \rightarrow (1)$

$$A = \begin{bmatrix} 17 & -15 \\ -15 & 17 \end{bmatrix}$$

$C_1 = 17 + 17 = 34$

$C_2 = 289 - 225 = 64$

The characteristic equation is,

$\lambda^2 - C_1\lambda + C_2 = 0$

$\lambda^2 - 34\lambda + 64 = 0$

$\lambda = 32, 2$

To find the eigen vectors :

$(A - \lambda I)X = 0$

$$\begin{pmatrix} 17-\lambda & -15 \\ -15 & 17-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow (2)$$

Case (i): $\lambda = 32$

$$② \Rightarrow \begin{pmatrix} 17-32 & -15 \\ -15 & 17-32 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -15 & -15 \\ -15 & -15 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-15x_1 - 15x_2 = 0$$

$$-15x_1 = 15x_2$$

$$-x_1 = x_2$$

$$\frac{x_1}{1} = \frac{x_2}{-1}$$

$$\therefore X_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Case (ii): $\lambda = 2$

$$② \Rightarrow \begin{pmatrix} 17-2 & -15 \\ -15 & 17-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 15 & -15 \\ -15 & 15 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$15x_1 - 15x_2 = 0$$

$$15x_1 = 15x_2$$

$$x_1 = x_2$$

$$\frac{x_1}{1} = \frac{x_2}{1}$$

$$X_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The modal matrix,

$$M = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

The normalized matrix,

$$N = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$N^T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$N^T A N = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 17 & -15 \\ -15 & 17 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 32 & 0 \\ 0 & 2 \end{pmatrix} = D$$

$$\therefore \; N^T A N = D$$

Now $Y^T D Y$ =

$$= (y_1 \;\; y_2) \begin{pmatrix} 32 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$= 32 y_1^2 + 2 y_2^2$$

Given : $Q = 128 \Rightarrow 32 y_1^2 + 2 y_2^2 = 128$

$$\frac{32 y_1^2}{128} + \frac{2 y_2^2}{128} = 1$$

$$\frac{y_1^2}{4} + \frac{y_2^2}{64} = 1$$

$\Rightarrow \dfrac{y_1^2}{2^2} + \dfrac{y_2^2}{8^2} = 1$ which represents the ellipse.

(ii) $Q = 14y_1^2 - 8y_2^2 = 0$ which represent pair of straight lines

(iii) $Q = 7y_1^2 - 5y_2^2 = 70$ which is a hyperbola.