UNIT II

ORTHOGONAL TRANSFORMATION OF A REAL SYMMETRIC MATRIX

Quadratic form :

A homogeneous polynomial of the second degree in any number of variables is called quadratic form.

Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ & $X^T = (x_1 \quad x_2 \quad x_3)$

$Q = X^T A X$

$$Q = (x_1 \quad x_2 \quad x_3) \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$Q = a_{11} x_1^2 + a_{22} x_2^2 + a_{33} x_3^2 + 2a_{12} x_1 x_2 + 2a_{23} x_2 x_3 + 2a_{31} x_3 x_1 \longrightarrow (1)$$

Here $a_{21} = a_{12}$, $a_{31} = a_{13}$, $a_{23} = a_{32}$

Equation (1) is called the matrix of the quadratic form.

Note:

$$Q = \begin{bmatrix} \text{Coef of } x^2 & \frac{1}{2}\text{coef of } xy & \frac{1}{2}\text{ coef } xz \\ \frac{1}{2}\text{ coef of } yz & \text{coef of } y^2 & \frac{1}{2}\text{ coef of } yz \\ \frac{1}{2}\text{ coef of } zx & \frac{1}{2}\text{ coef of } zy & \text{coef of } z^2 \end{bmatrix}$$

Nature of the quadratic form:

Let $Q = X^T A X$ be a quadratic form in n variables $x_1, x_2, \cdots x_n$

(i) Rank : Number of non-zero eigen values

(ii) Index : Number of positive square terms in the canonical form.

(iii) Signature : Difference between the number of positive and negative squares terms in the canonical form.

(iv) Nature :

Positive Definite : If all the eigen values are positive.

Positive Semi definite : If all the eigen values are positive and atleast one eigen value is zero.

Negative Definite : If all the eigen values are negative.

Negative Semi definite : If all the eigen values are negative and atleast one eigen value is zero.

Indefinite : If it has both positive and negative eigen values.

Problems :

(1) Find the matrix of the quadratic form $2x^2 + 3y^2 + 2z^2 + 2xy$.

Soln:

$$Q = \begin{bmatrix} \text{coef of } x^2 & \frac{1}{2}\text{ coef of } xy & \frac{1}{2}\text{ coef of } xz \\ \frac{1}{2}\text{ coef of } yx & \text{coef of } y^2 & \frac{1}{2}\text{ coef of } yz \\ \frac{1}{2}\text{ coef of } zx & \frac{1}{2}\text{ coef of } zy & \text{coef of } z^2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(2) Write the matrix of the quadratic form.

(i) $x^2 + 2y^2$

Soln:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(ii) $6x^2 + 3y^2 + 3z^2 - 4xy - 2yz + 4xz$

Soln:

$$\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$$