UNIT 5 Z - Transforms and Difference equations
Convolution Theorem

Inverse Z-transform:

If $Z[f(n)] = F(z)$, then $f(n)$ is called the inverse Z-transform of $F(z)$ and is denoted by

$$f(n) = Z^{-1}\{F(z)\}$$

Convolution Theorem:

If $F(z)$ and $G(z)$ are the Z-transforms of $f(n)$ and $g(n)$ respectively then

$$Z\{f(n) * g(n)\} = F(z)\,G(z)$$

where $f(n) * g(n)$ is defined as the convolution of $f(n)$ and $g(n)$ given by $f(n) * g(n) = \sum_{k=0}^{n} f(k)\,g(n-k)$

Proof:-

We have $F(z)G(z) = \left[\sum_{n=0}^{\infty} f(n)\,z^{-n}\right]\left[\sum_{n=0}^{\infty} g(n)\,z^{-n}\right]$

$$F(z)G(z) = \left[f(0) + f(1)z^{-1} + \dots + f(n)z^{-n} + \dots\right]\left[g(0) + g(1)z^{-1} + \dots + g(n)z^{-n}\right]$$

$$= \sum_{n=0}^{\infty}\left[f(0)g(n) + f(1)g(n-1) + \dots + f(n)g(0)\right]z^{-n}$$

$$= \sum_{n=0}^{\infty} A_n z^{-n}$$

Where $A_n = f(0)g(n) + \dots + f(n)g(0)$

$$= \sum_{k=0}^{n} f(k)\,g(n-k) = f(n) * g(n)$$

UNIT 5 Z - Transforms and Difference equations
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$$F(z)\,G(z) = \sum_{n=0}^{\infty} A_n z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(f(n) * g(n)\right) z^{-n}$$

$$F(z)\,G(z) = Z\left\{f(n) * g(n)\right\}$$

Inverse z-transform using convolution theorem.

1. Using convolution theorem, find $Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right]$

WKT $Z\left[a^n\right] = \frac{z}{z-a} \Rightarrow Z^{-1}\left[\frac{z}{z-a}\right] = a^n$

Let $F(z) = \frac{z}{z-a}$ and $G(z) = \frac{z}{z-b}$

$$\therefore\ Z^{-1}\left[F(z)\,G(z)\right] = Z^{-1}\left[F(z)\right] * Z^{-1}\left[G(z)\right]$$

$$= Z^{-1}\left[\frac{z}{z-a}\right] * Z^{-1}\left[\frac{z}{z-b}\right]$$

$$Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = a^n * b^n$$

$$= \sum_{k=0}^{n} a^k\, b^{n-k}$$

$$= b^n \sum_{k=0}^{n} a^k\, b^{-k}$$

$$= b^n \sum_{k=0}^{n} \left(\frac{a}{b}\right)^k$$

$$= b^n\left[1 + \left(\frac{a}{b}\right) + \left(\frac{a}{b}\right)^2 + \dots + \left(\frac{a}{b}\right)^n\right]$$

$$= b^n\left[\frac{-1 + \left(\frac{a}{b}\right)^{n+1}}{\left(\frac{a}{b}\right) - 1}\right] = b^n\left[\frac{\frac{a^{n+1} - b^{n+1}}{b^{n+1}}}{\frac{a-b}{b}}\right]$$

$$= \frac{a^{n+1} - b^{n+1}}{a-b}$$

UNIT 5 Z - Transforms and Difference equations
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Problems on convolution

3. find the z-transform of $f(n) * g(n)$ where $f(n) = \left(\frac{1}{2}\right)^n$ and $g(n) = \cos n\pi$ using convolution theorem.

By convolution theorem,

$$Z[f(n) * g(n)] = Z[f(n)] \cdot Z[g(n)] \rightarrow (1)$$

$$Z[f(n)] = Z\left[\left(\frac{1}{2}\right)^n\right] = \frac{z}{z - \frac{1}{2}} = \frac{2z}{2z-1}$$

and
$$Z[g(n)] = Z[\cos n\pi]$$
$$= Z[(-1)^n] = \frac{z}{z-(-1)} = \frac{z}{z+1}$$

$$(1) \Rightarrow \quad Z[f(n) * g(n)] = \frac{2z}{2z-1}\left(\frac{z}{z+1}\right)$$

$$= \frac{2z^2}{(2z-1)(z+1)}$$

4. Using Convolution theorem, find $z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right]$

$$z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right] = z^{-1}\left(\frac{z}{(z-1)} \cdot \frac{z}{z-3}\right)$$

$$= z^{-1}\left(\frac{z}{z-1}\right) * z^{-1}\left(\frac{z}{z-3}\right)$$

$$= (1)^n * 3^n$$

$$= \sum_{k=0}^{n} (1)^k \cdot 3^{n-k}$$

$$= \sum_{k=0}^{n} 3^{n-k}$$

$$= 3^n + 3^{n-1} + 3^{n-2} + 3^{n-3} + \dots + 3^0$$

$$= 1 + 3 + \dots + 3^{n-2} + 3^{n-1} + 3^{n}$$

$$= \frac{3^{n+1} - 1}{3 - 1}$$

$$= \frac{3^{n+1} - 1}{2}.$$

5. Using convolution theorem, find $Z^{-1}\left[\frac{8z^2}{(2z-1)(4z+1)}\right]$

$$Z^{-1}\left[\frac{8z^2}{(2z-1)(4z+1)}\right] = Z^{-1}\left[\frac{8z^2}{2(z-1/2)\cdot 4(z+1/4)}\right]$$

$$= Z^{-1}\left[\frac{z^2}{(z-1/2)(z+1/4)}\right]$$

$$= Z^{-1}\left[\frac{z}{(z-\frac{1}{2})}\cdot\frac{z}{(z+\frac{1}{4})}\right]$$

$$= Z^{-1}\left(\frac{z}{z-\frac{1}{2}}\right) * Z^{-1}\left(\frac{z}{z+\frac{1}{4}}\right)$$

$$= \left(\frac{1}{2}\right)^n * \left(\frac{-1}{4}\right)^n$$

$$= \sum_{K=0}^{n} \left(\frac{1}{2}\right)^K \left(\frac{-1}{4}\right)^{n-K}$$

$$= \left(\frac{-1}{4}\right)^n \sum_{K=0}^{n} \left(\frac{1}{2}\right)^K (-4)^K$$

$$= \left(\frac{-1}{4}\right)^n \sum_{K=0}^{n} \left[(-4)\left(\frac{+1}{2}\right)\right]^K = \left(\frac{-1}{4}\right)^n \sum_{K=0}^{n} (-2)^K$$

$$= \left(\frac{-1}{4}\right)^n \left[1 + (-2) + (-2)^2 + \dots + (-2)^n\right]$$

$$= \left(\frac{-1}{4}\right)^n \left[\frac{1-(-2)^{n+1}}{1-(-2)}\right] = \left(\frac{-1}{4}\right)^n \left[\frac{1+2(-2)^n}{3}\right]$$

UNIT 5 Z - Transforms and Difference equations
Convolution Theorem

6. Using Convolution theorem, find $Z^{-1}\left[\frac{z^2}{(z-4)(z-3)}\right]$

$$Z^{-1}\left[\frac{z^2}{(z-4)(z-3)}\right] = Z^{-1}\left[\frac{z}{z-4}\cdot\frac{z}{z-3}\right]$$

$$= Z^{-1}\left[\frac{z}{z-4}\right] * Z^{-1}\left[\frac{z}{z-3}\right]$$

$$= 4^n * 3^n$$

$$= \sum_{K=0}^{n} 4^K \cdot 3^{n-K}$$

$$= 3^n \sum_{K=0}^{n} 4^K \cdot 3^{-K} = 3^n \sum_{K=0}^{n}\left(\frac{4}{3}\right)^K$$

$$= 3^n\left[\left(\frac{4}{3}\right)^0 + \left(\frac{4}{3}\right)^1 + \left(\frac{4}{3}\right)^2 + \dots + \left(\frac{4}{3}\right)^n\right]$$

$$= 3^n\left[1 + \frac{4}{3} + \left(\frac{4}{3}\right)^2 + \dots + \left(\frac{4}{3}\right)^n\right]$$

$$= 3^n\left[\frac{(4/3)^{n+1} - 1}{4/3 - 1}\right]$$

7) Use Convolution, find $Z^{-1}\left[\frac{z^2}{(z-a)^2}\right]$

$$Z^{-1}\left[\frac{z^2}{(z-a)^2}\right] = Z^{-1}\left[\frac{z}{z-a}\cdot\frac{z}{z-a}\right] = Z^{-1}\left[\frac{z}{z-a}\right] * Z^{-1}\left[\frac{z}{z-a}\right]$$

$$= a^n * a^n$$

$$= \sum_{K=0}^{n} a^K \cdot a^{n-K}$$

$$= \sum_{K=0}^{n} a^n$$

$$= (n+1)a^n.$$