(1) Find the Fourier transform of the func. $f(x) = \begin{cases} a^2 - x^2, & |x| \le a \\ 0, & |x| > a \end{cases}$

and deduce that $\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right) \cos\frac{s}{2}\, ds = \frac{3\pi}{16}$

and $\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \frac{\pi}{15}$

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a^2 - x^2)\, (e^{isx})\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} a^2 e^{isx} dx - \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} x^2 e^{isx} dx$$

$$= \frac{a^2}{\sqrt{2\pi}} \left[\frac{e^{isx}}{is}\right]_{-a}^{a} - \frac{1}{\sqrt{2\pi}} \left[x^2 \frac{e^{isx}}{is} - 2x \frac{e^{isx}}{(is)^2} + \frac{2e^{isx}}{(is)^3}\right]_{-a}^{a}$$

$$= \frac{a^2}{\sqrt{2\pi}} \left[\frac{e^{isa} - e^{-isa}}{is}\right] - \frac{1}{\sqrt{2\pi}} \left[\frac{a^2 e^{isa}}{is} - \frac{2a e^{isa}}{(is)^2} + \frac{2e^{isa}}{(is)^3} - \right.$$

$$\left.\left[a^2 \frac{e^{-isa}}{is} + \frac{2a e^{-isa}}{(is)^2} + \frac{2e^{-isa}}{(is)^3}\right]\right]$$

$$= \frac{a^2}{\sqrt{2\pi}} \frac{2i \sin sa}{is} - \frac{1}{\sqrt{2\pi}} \left[\frac{a^2}{is} [e^{isa} - e^{-isa}] + \frac{(-2a)}{(is)^2}\right.$$

$$\left.[e^{isa} + e^{-isa}] + \frac{(+2)}{(is)^3} [e^{isa} - e^{-isa}]\right]$$

$$= \frac{a^2}{\sqrt{2\pi}} \frac{2i\sin sa}{is} - \frac{a^2\, 2i \sin sa}{\sqrt{2\pi}\, is} + \frac{2a}{\sqrt{2\pi}} \frac{\overset{2\cos sa}{\cancel{2i \sin sa}}}{(is)^2} -$$

$$+ \frac{2}{\sqrt{2\pi}} \frac{2i \sin sa}{(is)^3}$$

$$= \frac{-2a}{\sqrt{2\pi}} \frac{2\cos sa}{s^2} - \frac{-2}{\sqrt{2\pi}} \frac{2 i \sin sa}{i s^3}$$

$$= -\frac{4a}{\sqrt{2\pi}} \frac{\cos sa}{s^2} + \frac{4}{\sqrt{2\pi}} \frac{\sin sa}{s^3}$$

$$F(s) = \frac{4}{\sqrt{2\pi}} \left[\frac{\sin sa - as\cos sa}{s^3} \right].$$

Inversion formula: $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{4}{\sqrt{2\pi}} \left[\frac{\sin sa - as\cos sa}{s^3} \right] e^{-isx} ds.$$

$$a^2 - x^2 = \frac{4}{2\pi} \int_{-\infty}^{\infty} \left[\frac{\sin sa - as\cos sa}{s^3} \right] \cos sx \, ds$$

Take $x = 1/2$ and put $a = 1$.

$$1^2 - (1/2)^2 = \frac{2}{\pi} \cdot 2 \int_0^{\infty} \left[\frac{\sin s - s\cos s}{s^3} \right] \cos \frac{s}{2} \, ds.$$

$$1 - 1/4 = \frac{4}{\pi} \int_0^{\infty} \left[\frac{\sin s - s\cos s}{s^3} \right] \cos s/2 \, ds.$$

$$\frac{3}{4} \times \frac{\pi}{4} \left(\frac{3\pi}{16}\right) = \int_0^{\infty} \left[\frac{\sin s - s\cos s}{s^3} \right] \cos s/2 \, ds.$$

Parseval's Identity:

$$\int_{-\infty}^{\infty} (f(x))^2\,dx = \int_{-\infty}^{\infty} (F(s))^2\,ds$$

$$\int_{-a}^{a} (a^2-x^2)^2\,dx = \int_{-\infty}^{\infty} \left[\frac{4}{\sqrt{2\pi}}\left[\frac{\sin sa - as\cos sa}{s^3}\right]\right]^2 ds$$

put $a=1$,

$$\int_{-1}^{1} (1-x^2)^2\,dx = \frac{16}{2\pi}\times 2\int_{0}^{\infty}\left[\frac{\sin sa - s\cos s}{s^3}\right]^2 ds$$

$$2\int_{0}^{1} (1-x^2)^2\,dx = \frac{16}{\pi}\int_{0}^{\infty}\left[\frac{\sin s - s\cos s}{s^3}\right]^2 ds$$

$$= \frac{16}{\pi}\int_{0}^{\infty}\left[\frac{\sin s - s\cos s}{s^3}\right]^2 ds$$

$$\frac{16}{15} = \frac{16}{\pi}\int_{0}\left[\frac{\sin s - s\cos s}{s^3}\right]^2 ds$$

$$\frac{\pi}{15} = \int_{0}^{\infty}\left[\frac{\sin s - s\cos s}{s^3}\right]^2 ds$$

(2) Find the fourier transform & deduce that $\int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right) dt$ & $\int_0^\infty \left[\frac{\sin t - t\cos t}{t^3}\right]^2 dt$ for

(i) $f(x) = \begin{cases} 1-x^2, & |x|<1 \\ 0, & |x|>1 \end{cases}$ (ii) $f(x) = \begin{cases} 2^2-x^2, & |x|<2 \\ 0, & |x|>2. \end{cases}$

(1) Find the fourier transform of $f(x) = e^{-a^2x^2}$.

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx} dx$$

$\int_{-\infty} e^{-a^2x^2} \cos sx \, d$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2+isx} dx$$

$\int e^{-a^2x^2} \frac{\sin sx}{s}$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-[a^2x^2-isx]} dx$$

$+\left(\frac{\cos sx}{s^2}\right)+$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2\left[x^2-\frac{isx}{a^2}+\left(\frac{is}{2a^2}\right)^2-\left(\frac{is}{2a^2}\right)^2\right]} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2\left\{\left[x-\frac{is}{2a^2}\right]^2-\left(\frac{is}{2a^2}\right)^2\right\}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left\{\left[ax-\frac{is}{2a}\right]^2-\frac{\cancel{a^2}i^2s^2}{4a^{\cancel{4}2}}\right\}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[ax - \frac{is}{2a}\right]^2 - \frac{s^2}{4a^2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[ax - \frac{is}{2a}\right]^2} e^{-\frac{s^2}{4a^2}} dx$$

$$= e^{-\frac{s^2}{4a^2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[ax - \frac{is}{2a}\right]^2} dx.$$

put $t = ax - \frac{is}{2a}$

$$dt = a\,dx \Rightarrow dx = \frac{dt}{a}.$$

$$= e^{-\frac{s^2}{4a^2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2} \frac{dt}{a}$$

$$= e^{-s^2/4a^2} \frac{1}{a\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2} dt$$

$$\boxed{\begin{array}{l} \text{Result: } \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \\ \int_{0}^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \end{array}}$$

$$= e^{-\frac{s^2}{4a^2}} \frac{1}{a\sqrt{2\pi}} \cdot \sqrt{\pi} = e^{-\frac{s^2}{4a^2}} \cdot \frac{1}{a\sqrt{2}}.$$