PROPERTIES OF FOURIER TRANSFORMS :

1) LINEAR PROPERTY :

Show that the operator 'F' is linear

(a) $F[af(x)+bg(x)] = aF[f(x)] + bF[g(x)]$

Now $F[af(x)+bg(x)]$

W.K.T $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$

$$F[af(x)+bg(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [af(x)e^{isx}] + [bg(x)]e^{isx} dx .$$

$$= \frac{a}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx + \frac{b}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{isx} dx$$

$$= aF[f(x)] + bF[g(x)]$$, F is linear.

III^{ly} $F_s[af(x)+bg(x)] = aF_s[f(x)] + bF_s[g(x)]$

$F_c[af(x)+bg(x)] = aF_c[f(x)] + bF_c[g(x)]$

2) SHIFTING PROPERTY :

(i) $F[f(x-a)] = e^{isa} F(s)$

(ii) $F[e^{iax} f(x)] = F(s+a)$

(i) $F[f(x-a)] = e^{isa} F(s)$

WKT $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx} dx$

Now $F[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a)\, e^{isx} dx$

put $x-a = p \Rightarrow dx = dp$.

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(p)\, e^{is(a+p)} dp$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(p)\, e^{isa} \cdot e^{isp} dp$$

$$= e^{isa} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(p)\, e^{isp} dp$$

$$= e^{isa} F(s)$$

(ii) $F[e^{iax} f(x)] = F(s+a)$

WKT $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$

Now $F[e^{iax} f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax} f(x) e^{isx} dx$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(s+a)x} f(x)\, dx$

$= F(s+a)$.

3> CHANGE OF SCALE PROPERTY:

$F[f(ax)] = \frac{1}{a} F\left(\frac{s}{a}\right), \; a>0$

WKT $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$

$F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax) e^{isx} dx$

put $t = ax \Rightarrow dt = a\, dx$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ist/a} \frac{dt}{a}$

$= \frac{1}{a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ist/a} dt$

$= \frac{1}{a} F\left(\frac{s}{a}\right)$.

4) $F[x^n f(x)] = (-i)^n \frac{d^n F}{ds^n}$

5) MODULATION (PROPERTY) THEOREM:

(i) $F[f(x)\cos ax] = \frac{1}{2}[F(s+a) + F(s-a)]$

(ii) $F_s[f(x)\cos ax] = \frac{1}{2}[F_s(s+a) + F_s(s-a)]$

(iii) $F_c[f(x)\cos ax] = \frac{1}{2}[F_c(a+s) + F_c(a-s)]$

(iv) $F_s[f(x)\sin ax] = \frac{1}{2}[F_c(s-a) - F_c(s+a)]$

(v) $F_c[f(x)\sin ax] = \frac{1}{2}[F_s(a+s) + F_s(a-s)]$

6) (i) $F_c[x f(x)] = \frac{d}{ds} F_s[f(x)]$

(ii) $F_s[x f(x)] = -\frac{d}{ds}[F_c[f(x)]]$

7) (i) $F[f'(x)] = -isF(s)$.

(ii) $F[f^n(x)] = (-i)^n s^n F(s)$.

8) $F\left[\int_a^x f(x)\,dx\right] = \frac{F(s)}{(-is)}$

9) (i) $F[\overline{f(x)}] = \overline{F(-s)}$

(ii) $F[\overline{f(-x)}] = \overline{F(s)}$.

Find Fourier cosine & sine transform of xe^{-ax}

Soln:

WKT $F_s[x f(x)] = -\frac{d}{ds} F_c[f(x)]$

Given: $xf(x) = xe^{-ax}$; here $f(x) = e^{-ax}$.

Now. $F_s[xe^{-ax}] = -\frac{d}{ds} F_c[e^{-ax}]$

$$= -\frac{d}{ds}\left[\sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx\right]$$

$$= -\frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+s^2}\right]\right]$$

$$= \cdot \sqrt{\frac{2}{\pi}} \frac{a}{(a^2+s^2)^2} \cdot (-2s)$$

$$F_s[xe^{-ax}] = \sqrt{\frac{2}{\pi}} \cdot \frac{2as}{(a^2+s^2)^2}$$

Now $F_c[xe^{-ax}] = \frac{d}{ds} F_s[e^{-ax}]$

$$= \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx \, dx\right]$$

$$= \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\left[\frac{s}{s^2+a^2}\right]\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{(s^2+a^2)(1) - s\cdot(2s)}{(s^2+a^2)^2}\right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{a^2 - s^2}{(s^2+a^2)^2}$$