Using parseval's Identity calculate $\int_0^\infty \frac{x^2}{(a^2+x^2)^2}\,dx$, if $a>$

Soln:

WKT $F_s(s) = F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx = \sqrt{\frac{2}{\pi}}\left[\frac{s}{a^2+s^2}\right]$

parseval's Identity:

$$\int_0^\infty (f(x))^2\,dx = \int_0^\infty [F_s(s)]^2\,ds$$

Here $f(x) = e^{-ax}$

$$\int_0^\infty (e^{-ax})^2\,dx = \int_0^\infty \left[\sqrt{\frac{2}{\pi}}\left[\frac{s}{a^2+s^2}\right]\right]^2 ds$$

$$\int_0^\infty e^{-2ax}\,dx = \frac{2}{\pi}\int_0^\infty \left(\frac{s}{a^2+s^2}\right)^2 ds$$

$$\left[\frac{e^{-2ax}}{-2a}\right]_0^\infty = \frac{2}{\pi}\int_0^\infty \left(\frac{s}{a^2+s^2}\right)^2 ds$$

$$\frac{1}{2a}\cdot\frac{\pi}{2} = \int_0^\infty \left[\frac{s}{(s^2+a^2)}\right]^2 ds$$

put $s=x$

$$\frac{\pi}{4a} = \int_0^\infty \left[\frac{x}{x^2+a^2}\right]^2 dx.$$

2) Evaluate $\int_0^\infty \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx$ using transforms.

Soln:

WKT $F_s(s) = F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx\, dx = \sqrt{\frac{2}{\pi}} \left[\frac{s}{a^2+s^2}\right]$

$G_s(s) = F_s[g(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-bx} \sin sx\, dx = \sqrt{\frac{2}{\pi}} \left[\frac{s}{b^2+s^2}\right]$

Parseval's Identity:

$$\int_0^\infty f(x)\cdot g(x)\,dx = \int_0^\infty F_s(s)\, G_s(s)\,ds$$

Here $f(x) = e^{-ax}$, $g(x) = e^{-bx}$

$$\int_0^\infty e^{-ax}\cdot e^{-bx}\,dx = \int_0^\infty F_s[f(x)]\, F_s[g(x)]\,ds$$

$$\int_0^\infty e^{-(a+b)x}\,dx = \int_0^\infty \sqrt{\frac{2}{\pi}}\left[\frac{s}{s^2+a^2}\right] * \sqrt{\frac{2}{\pi}}\left[\frac{s}{s^2+b^2}\right] ds$$

$$\left.\frac{e^{-(a+b)}}{-(a+b)}\right]_0^\infty = \frac{2}{\pi}\int_0^\infty \frac{s^2}{(s^2+a^2)(s^2+b^2)}\,ds$$

$$\frac{1}{a+b}\cdot\frac{\pi}{2} = \int_0^\infty \frac{s^2}{(s^2+a^2)(s^2+b^2)}\,ds$$

put $s = x$

$$\frac{\pi}{2(a+b)} = \int_0^\infty \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx.$$

Using transforms evaluate:

(i) $\int_0^\infty \frac{dx}{(x^2+a^2)^2}$

(ii) $\int_0^\infty \frac{ds}{(s^2+4)(s^2+1)}$

(iii) $\int_0^\infty \frac{x^2}{(x^2+9)(x^2+16)}\,dx$