Lattices and Boolean Algebra

Relation:

Relation is called a binary operation between a pair of objects.

Properties of Relation:

Reflexive:

Let R be a relation on set X

R is reflexive if xRx, $\forall x \in X$.

Symmetric:

R is symmetric if $xRy \Rightarrow yRx$, $\forall x, y \in X$.

Transitive:

R is Transitive if xRy and $yRz \Rightarrow xRz$,

$\forall x, y, z \in X$.

Antisymmetric:

R is antisymmetric if xRy and $yRx \Rightarrow x = y$

Eg:

Let $A = \{1, 2, 3\}$ and R be the relation "$\leq$"

$R = \{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$

∴ R is reflexive, Transitive, Antisymmetric.

Equivalence Relation:

A relation R on X is called an equivalence relation on X if R satisfies reflexive, symmetric and Transitive.

Partial order Relation:

A relation R which satisfies reflexive, antisymmetric and transitive is called an partial order relation.

Partially ordered set or Posets

A set together with a partial order relation defined on it is called partially ordered set. It is denoted by $(P, \leq)$

Hasse diagram:

A graphical representation of a poset is called Hasse diagram.

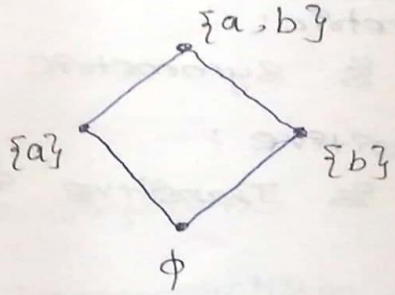
1]. Draw Hasse diagram for $(P(A), \subseteq)$ where

i). $A = \{a, b\}$ ii). $A = \{a, b, c\}$

Soln:

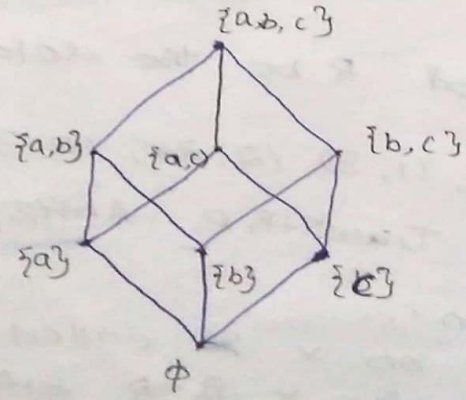
i). $A = \{a, b\}$

$P(A) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$



ii). $A = \{a, b, c\}$

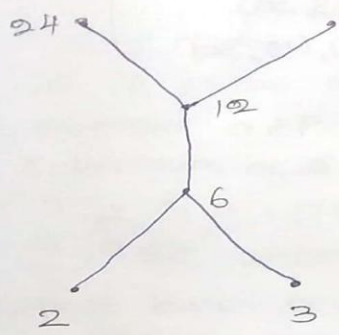
$P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$



2] If $X = \{2, 3, 6, 12, 24, 36\}$ and the relation "$\leq$" is defined as $a \leq b$ if a/b. Draw the Hasse diagram of $(X, \leq)$.

Soln.

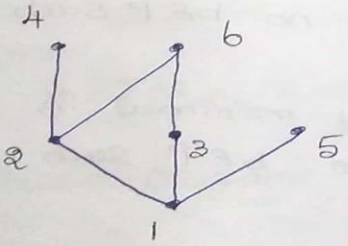
$R = \{(2,6), (2,12), (2,24), (2,36), (3,6), (3,12), (3,24), (3,36), (6,12), (6,24), (6,36), (12,24), (12,36)\}$



3. Draw Hasse diagram where $X = \{1,2,3,4,5,6\}$ and "/" is a partial ordering relation on X.

Soln.

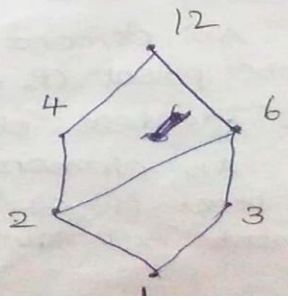
$R = \{(1,2), (1,3), (1,4), (1,5), (1,6), (2,4), (2,6), (3,6)\}$



4. If $X = \{1,2,3,4,6,12\}$ and "$\leq$" is the relation such that $x < y$ iff x/y.

Soln.

$R = \{(1,2), (1,3), (1,4), (1,6), (1,12), (2,4), (2,6), (2,12), (3,6), (3,12), (4,12), (6,12)\}$



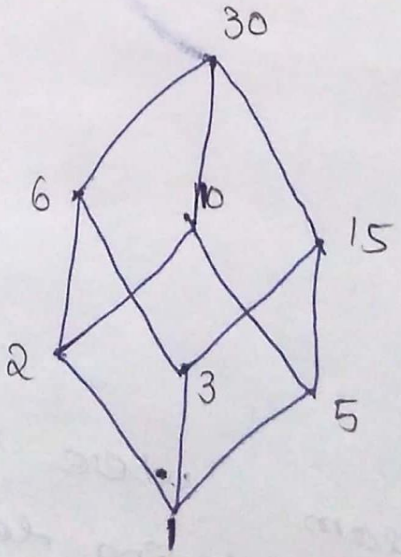
5]. Draw the Hasse Diagram for D_{30} (Divisors of 30)

$(P_{30}, /)$

Soln.

$D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$

$R = \{(1,2), (1,3), (1,5), (1,6), (1,10), (1,15), (1,30)$
$(2,6), (2,10), (2,30), (3,6), (3,15), (3,30)$
$(5,10), (5,15), (5,30), (6,30), (10,30), (15,30)\}$.



Maximal and Minimal Element:

An element $a \in P$ is called maximal in the poset $(P, \leq)$ if there is no $b \in P$ such that $b > a$.

An element $a \in P$ is called minimal in the Poset $(P, \leq)$ if there is no $b \in P$ such that $b < a$.

Greatest and Least element:

An element 'a' is the greatest element of the poset $(P, \leq)$ if $b \leq a$ for all $b \in P$. The greatest element is unique if it exist.

An element 'a' is the least element of the poset $(P, \leq)$ if $a \leq b$ for all $b \in P$. The least element is unique if it exist.

Upper bound and Lower Bound

Let $(P, \leq)$ be a poset and $A \subseteq P$.

Any elt $x \in P$ is an upper bound of A if for all $a \in A$ and $a \leq x$.

Similarly, an elt $x \in P$ is a lower bound of A if for $a \in A$ and $x \leq a$.

Least upper bound:

Let $(P, \leq)$ be a poset and $A \subseteq P$.

An element $a \in P$ is said to be least upper bound or supremum of A if

i) a is a upper bound of A

ii) $a \leq c$, where c is any other upper bound of A.

Greatest lower bound:

Let $(P, \leq)$ be a poset and $A \subseteq P$.

An element $b \in P$ is said to be greatest lower bound or infimum of A if

i) b is a lower bound of A

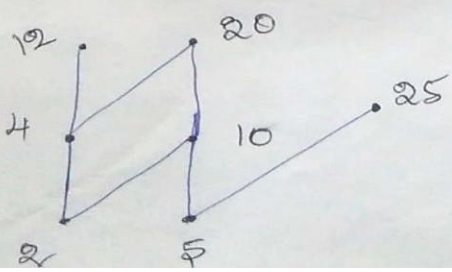
ii) $b \geq d$, where d is any other lower bound of A.

1. which elements of the poset $\{2, 4, 5, 10, 12, 20, 25\}$ are maximal and which are minimal?

Soln.

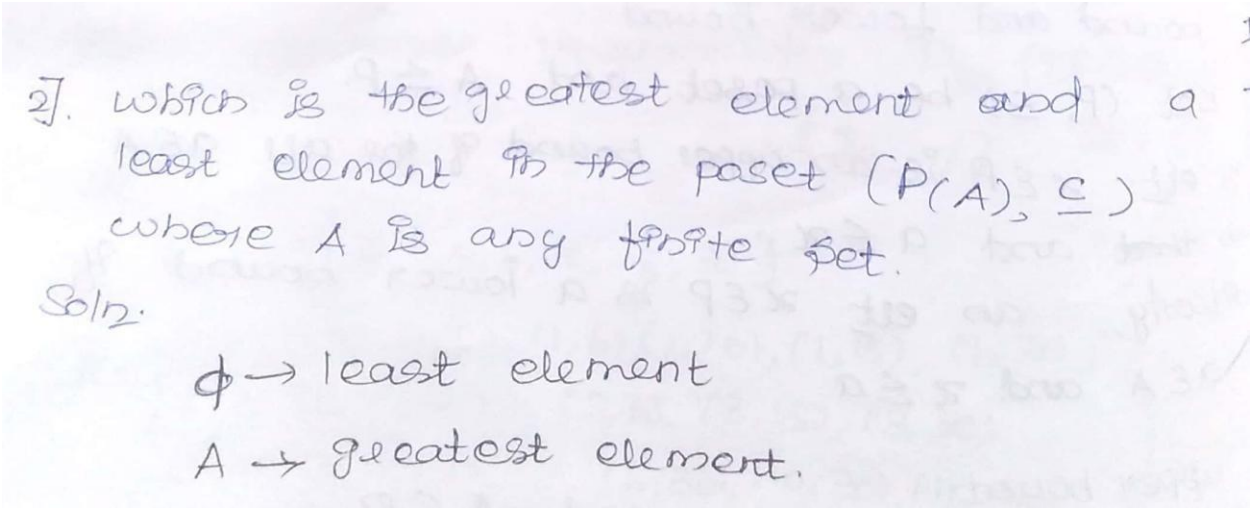
$R = \{(2,4), (2,10), (2,12), (2,20), (4,12), (4,20), (5,10), (5,20), (5,25), (10,20)\}$

Hasse Diagram:



maximal elements:
12, 20, 25

minimal elements:
2, 5



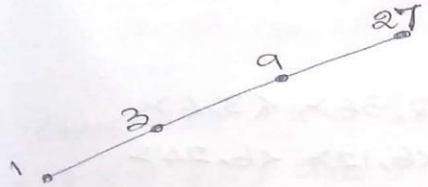
2]. Which is the greatest element and a least element in the poset $(P(A), \subseteq)$ where A is any finite set.

Soln.

$\phi \rightarrow$ least element

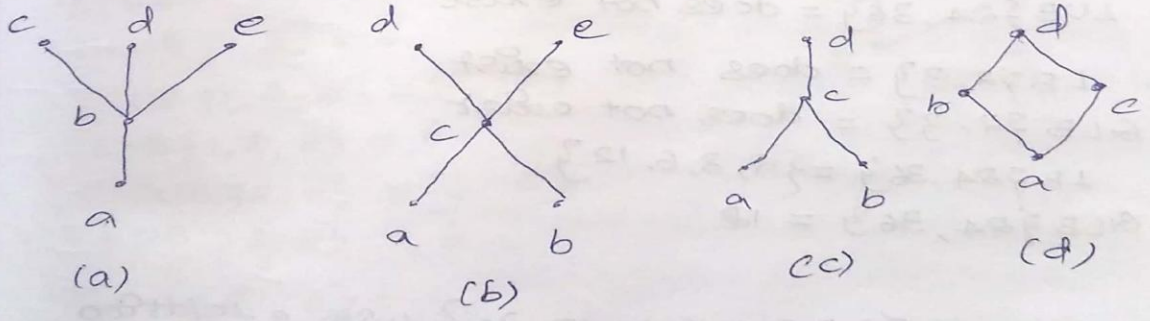
$A \rightarrow$ greatest element.

3. Determine whether the poset represented by the hasse diagram of the following have a least, greatest, maximal & minimal.



max elt. = 27
min. elt = 1
least elt. = 1
Greatest elt. = 27

1. Determine whether the posets represented by each of the HD in the following figure have a greatest elt. and a least elt.



Soln

(a) least elt. is a
greatest elt. does not exist

(b) neither a least nor a greatest elt.

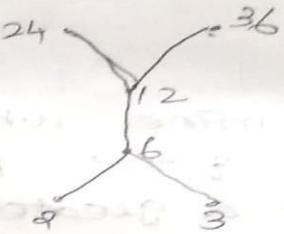
(c) least elt. does not exist
greatest elt. is d

(d) least elt. is a
greatest elt. is d.

5]. Consider $X = \{2, 3, 6, 12, 24, 36\}$ and $R = \{\langle a, b\rangle, a/b\}$. Find LUB and GLB of (2, 3) and (24, 36).

Soln.

$R = \{\langle 2,6\rangle \langle 2,12\rangle, \langle 2,24\rangle \langle 2,36\rangle \langle 3,6\rangle \langle 3,12\rangle \langle 3,24\rangle \langle 3,36\rangle \langle 6,12\rangle \langle 6,24\rangle \langle 6,36\rangle \langle 12,24\rangle \langle 12,36\rangle\}$



i). $UB\{2,3\} = \{6, 12, 24, 36\}$

$LUB\{2,3\} = 6$

$UB\{24,36\}$ = does not exist

$LUB\{24,36\}$ = does not exist

ii). $LB\{2,3\}$ = does not exist

$GLB\{2,3\}$ = does not exist

$LB\{24,36\} = \{2, 3, 6, 12\}$

$GLB\{24,36\} = 12$

6]. Let $D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$ with a relation $x \le y$ iff x divides y.

Find:

i). All lower bounds of 10 and 15.

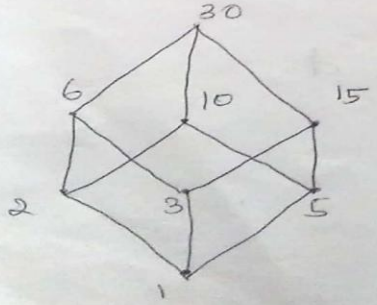
ii). GLB of 10 and 15

iii). All upper bound ~~of~~ 10 and 15

iv). LUB of 10 and 15

v). Draw Hasse Diagram for D_{30}.

Soln.

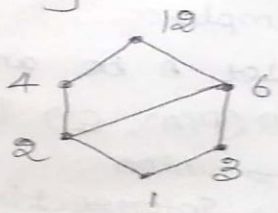


i). LB (10, 15) = {1, 5}
ii). GLB (10, 15) = 5
iii). UB (10, 15) = 30
LUB (10, 15) = 30

7]. Consider $X = \{1, 2, 3, 4, 6, 12\}$ and $R = \{\langle a, b\rangle / a|b\}$. Find LUB and GLB for the Poset (X, R).

Soln.

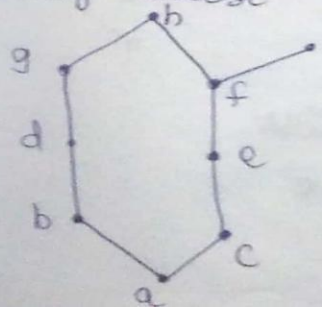
$R = \{(1,2), (1,3), (1,4), (1,6), (1,12), (2,4), (2,6), (2,12), (3,6), (3,12), (4,12), (6,12)\}$



i). UB {1, 3} = {3, 6, 12}
LUB {1, 3} = 3
UB {1, 2, 3} = {6, 12}
LUB {1, 2, 3} = 6
UB {2, 3} = {6, 12}
LUB {2, 3} = 6

ii).
LB {1, 3} = 1
GLB {1, 3} = 1
LB {1, 2, 3} = 1
GLB {1, 2, 3} = 1
LB {2, 3} = 1
GLB {2, 3} = 1

8]. Find the GLB & LUB of {b, d, g}, if they exist on the Poset given below.



Soln.

UB {b, d, g} = g, h
LUB {b, d, g} = g
LB {b, d, g} = a, b
GLB {b, d, g} = b