



UNIT 4- ALGEBRAIC STRUCTURES

Groups

Properties of Group:

i]. The identity element in a group is unique.

ii]. The inverse element in a group is unique

iii]. Cancellation law:

$$\forall a, b \in G \quad i). a * b = a * c \Rightarrow b = c \quad \text{[Left cancellation]}$$

$$ii). b * a = c * a \Rightarrow b = c \quad \text{[Right cancellation]}$$

iv]. Let G be a group.

$$\text{If } a, b \in G \text{ then } (a * b)^{-1} = b^{-1} * a^{-1}$$

Proof:

$$\text{Let } a, b \in G$$

and a^{-1}, b^{-1} be their inverses resply.

Let $(G, *)$ be a group.

$$\text{Let } a, b \in G.$$

$$a * a^{-1} = a^{-1} * a = e$$

$$b * b^{-1} = b^{-1} * b = e$$

$$\text{Now } (a * b) * (b^{-1} * a^{-1}) = a * (b * b^{-1}) * a^{-1} \quad \text{(Associative)}$$

$$= (a * e) * a^{-1}$$

$$= a * a^{-1} \quad \text{Identity}$$

$$(a * b) * (b^{-1} * a^{-1}) = e \rightarrow (1)$$

$$\text{Similarly } (b^{-1} * a^{-1}) * (a * b) = e \rightarrow (2)$$

$$\text{From (1) and (2), } (a * b)^{-1} = b^{-1} * a^{-1}$$

The inverse of $(a * b)$ is $b^{-1} * a^{-1}$

Hence proved.

v]. Prove that a group $(G, *)$ is abelian iff

$$(a * b)^2 = a^2 * b^2, \quad \forall a, b \in G$$

Proof:

Assume that G is abelian

$$\therefore a * b = b * a, \quad a, b \in G \rightarrow (1)$$



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Now $a^2 * b^2 = (a * a) * (b * b)$
 $= a * [a * (b * b)]$ Associative
 $= a * [(a * b) * b]$ Associative
 $= a * [(b * a) * b]$ Com By (1)
 $= (a * b) * (a * b)$ Associative
 $= (a * b)^2$

conversely,
 Assume that $(a * b)^2 = a^2 * b^2$
 $(a * b) * (a * b) = (a * a) * (b * b)$
 $a * [b * (a * b)] = a * [a * (b * b)]$
 $b * (a * b) = a * (b * b)$ left cancellation law
 $(b * a) * b = (a * b) * b$
 $b * a = a * b$ right cancellation law

$\Rightarrow G$ is abelian.

b). If $(G, *)$ is an abelian group, ST
 $(a * b)^n = a^n * b^n, \forall a, b \in G$, where n is a +ve integer
 proof:
 since $(G, *)$ is abelian, we've
 $a * b = b * a, \forall a, b \in G \rightarrow (1)$
 for $a, b \in G$, we've $(a * b)^1 = (b * a)^1$ by (1)
 and $(a * b)^2 = (a * b) * (a * b)$
 $= a * (b * a) * b$ Associative
 $= a * (a * b) * b$ by (1)
 $= (a * a) * (b * b)$ Associative
 $= a^2 * b^2$

Thus, the result is true for $n=1, 2, \dots$
 let us assume that the result is valid for
 $n=m$.



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$(a * b)^m = a^m * b^m \rightarrow (2)$
 Now $(a * b)^{m+1} = (a * b)^m * (a * b)$
 $= (a^m * b^m) * (a * b)$ By (2)
 $= a^m * (b^m * a) * b$ Associative
 $= a^m * (a * b^m) * b$ (G is abelian)
 $= (a^m * a) * (b^m * b)$
 $(a * b)^{m+1} = a^{m+1} * b^{m+1}$
 Hence by induction, the result is true for
 +ve integral values of n .

i]. In a group G , Prove that an elt. $a \in G$
 such that $a^2 = e$, $a \neq e$ iff $a = a^{-1}$

Proof:
 Assume that $a = a^{-1}$
 To prove: $a^2 = e$
 Now consider $a^2 = a * a$
 $= a * a^{-1}$
 $= e$

Conversely, Assume $a^2 = e$
 To prove: $a = a^{-1}$
 Now $a^2 = e$
 $a * a = e$
 $a^{-1} * (a * a) = a^{-1} * e$ [Premultiply by a^{-1}]
 $(a^{-1} * a) * a = a^{-1}$
 $e * a = a^{-1}$
 $a = a^{-1}$

ii]. If every elt. of a group G has its own
 inverse, then G is abelian.



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Proof:

Let $(G, *)$ be a group.

for $a, b \in G$, we've $a * b \in G$

Given $a = a^{-1}$ and $b = b^{-1}$

$$\begin{aligned} \text{Now } a * b &= (a * b)^{-1} && \text{[It has its own inverse]} \\ &= b^{-1} * a^{-1} && \text{(By property)} \\ &= b * a \end{aligned}$$

$\therefore G$ is abelian

Converse need not be true.