Theorem: 1

State and prove Demorgan's law of lattice.

(or)

If $(L, \wedge, \vee, 0, 1)$ is a complemented lattice, then prove that 1. $(a \wedge b)' = a' \vee b'$ (or) $\overline{(a \wedge b)} = \bar{a} \vee \bar{b}$

2. $(a \vee b)' = a' \wedge b'$ (or) $\overline{(a \vee b)} = \bar{a} \wedge \bar{b}$

Proof:

I. To prove that complement of $a \wedge b$ is $a' \vee b'$.

i.e., i). $(a \wedge b) \wedge (a' \vee b') = 0$

and ii) $(a \wedge b) \vee (a' \vee b') = 1$

Now i). $(a \wedge b) \wedge (a' \vee b')$

$= ((a \wedge b) \wedge a') \vee ((a \wedge b) \wedge b')$

$= ((b \wedge a) \wedge a') \vee (a \wedge (b \wedge b'))$

$= (b \wedge (a \wedge a')) \vee (a \wedge (b \wedge b'))$

$= (b \wedge 0) \vee (a \wedge 0)$

$= 0 \vee 0$

$= 0 \longrightarrow (1)$

ii). $(a \wedge b) \vee (a' \vee b') = (a \vee (a' \vee b')) \wedge (b \vee (a' \vee b'))$

$= ((a \vee a') \vee b') \wedge (b \vee (b' \vee a'))$

$= ((a \vee a') \vee b') \wedge ((b \vee b') \vee a')$

$= (1 \vee b') \wedge (1 \vee a')$

$= 1 \wedge 1$

$= 1 \longrightarrow (2)$

From (1) and (2),

$(a \wedge b)' = a' \vee b'$

2). To prove that complement of $a \vee b$ is $a' \wedge b'$.

i.e., i). $(a \vee b) \wedge (a' \wedge b') = 0$

ii). $(a \vee b) \vee (a' \wedge b') = 1$

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i). $(a \vee b) \wedge (a' \wedge b') = (a \wedge (a' \wedge b')) \vee (b \wedge (a' \wedge b'))$

$= ((a \wedge a') \wedge b') \vee (b \wedge (b' \wedge a'))$

$= ((a \wedge a') \wedge b') \vee ((b \wedge b') \wedge a')$

$= (0 \wedge b') \vee (0 \wedge a')$

$= 0 \vee 0$

$= 0 \longrightarrow (3)$

ii). $(a \vee b) \vee (a' \wedge b') = ((a \vee b) \vee a') \wedge ((a \vee b) \vee b')$

$= (a \vee (b \vee a')) \wedge (a \vee (b \vee b'))$

$= (a \vee (a' \vee b)) \wedge (a \vee (b \vee b'))$

$= ((a \vee a') \vee b) \wedge (a \vee (b \vee b'))$

$= (1 \vee b) \wedge (a \vee 1)$

$= 1 \wedge 1$

$= 1 \longrightarrow (4)$

From (3) and (4), $(a \vee b)' = a' \wedge b'$

Hence proved.

Theorem : 2

(4) Prove that in a complemented distributive lattice, complement is unique.

(or)

If $(L, \wedge, \vee, 0, 1)$ is a distributive lattice, then each elt. $x \in L$ has atmost one complement

Proof:

Let x and y be two complement of a.

Since x is a complement of a.

$$\left.\begin{array}{l} a \wedge x = 0 \\ a \vee x = 1 \end{array}\right\} \longrightarrow (1)$$

Since y is a complement of a.

$a \wedge y = 0$, $a \vee y = 1 \longrightarrow (2)$

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Now, $x = x \vee 0$ by (2)

$= x \vee (a \wedge y)$

$= (x \vee a) \wedge (x \vee y)$

$= (a \vee x) \wedge (x \vee y)$

$= 1 \wedge (x \vee y)$ by (1)

$x = x \vee y \rightarrow (A)$

and $y = y \vee 0$

$= y \vee (a \wedge x)$

$= (y \vee a) \wedge (y \vee x)$

$= (a \vee y) \wedge (x \vee y)$

$= 1 \wedge (x \vee y)$ by (2)

$y = x \vee y \longrightarrow (B)$

From (A) and (B), we've

$x = x \vee y = y$

$\therefore x = y$

Hence proved.

Problem:

1. In a complemented, distributive lattice, Show that the following are equivalent.

$a \leq b \Leftrightarrow a \wedge b' = 0 \Leftrightarrow a' \vee b = 1 \Leftrightarrow b' \leq a'$

(or)

In otherwords the following are equivalent

i). $a \leq b$ ii). $a \wedge b' = 0$ iii). $a' \vee b = 1$ iv). $b' \leq a'$

(or) $a \leq b \Leftrightarrow a * b' = 0 \Leftrightarrow a' \oplus b = 1 \Leftrightarrow b' \leq a'$

Proof:

To prove (i) $\Rightarrow$ (ii)

Let $a \leq b$.

Then $a \wedge b = a$ and $a \vee b = b \rightarrow (1)$

Now, $a \wedge b' = ((a \wedge b) \wedge b')$

$= (a \wedge (b \wedge b'))$

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$= (a \wedge 0)$ $\quad \because b \wedge b' = 0$

$= 0$ $\quad \because a \wedge 0 = 0$

Hence $a \leq b \Rightarrow a \wedge b' = 0$

$0 \wedge x = 0$
$0 \vee x = x$
$1 \wedge x = x$
$1 \vee x = 1$

(ii) $\Rightarrow$ (iii)

Let $a \wedge b' = 0$

Take complement on bothsides,

$(a \wedge b')' = 0'$

$a' \vee b = 1$

$\therefore a \wedge b' = 0 \Rightarrow a' \vee b = 1$

(iii) $\Rightarrow$ (iv)

Let $a' \vee b = 1$

$\Rightarrow (a' \vee b) \wedge b' = 1 \wedge b'$ (cancellation law)

$(a' \wedge b') \vee (b \wedge b') = b'$ (Distributive law)

$(a' \wedge b') \vee 0 = b'$ $\quad \because b \wedge b' = 0$

$a' \wedge b' = b'$

$\therefore b' \leq a'$

$\therefore a' \vee b = 1 \Rightarrow b' \leq a'$

(iv) $\Rightarrow$ (i)

Let $b' \leq a'$.

Then $a' \wedge b' = b'$

Take complement on bothsides,

$(a' \wedge b')' = (b')'$

$a \vee b = b$

$\Rightarrow b \geq a$ or $a \leq b$

$\therefore b' \leq a' \Rightarrow a \leq b$

Hence proved.