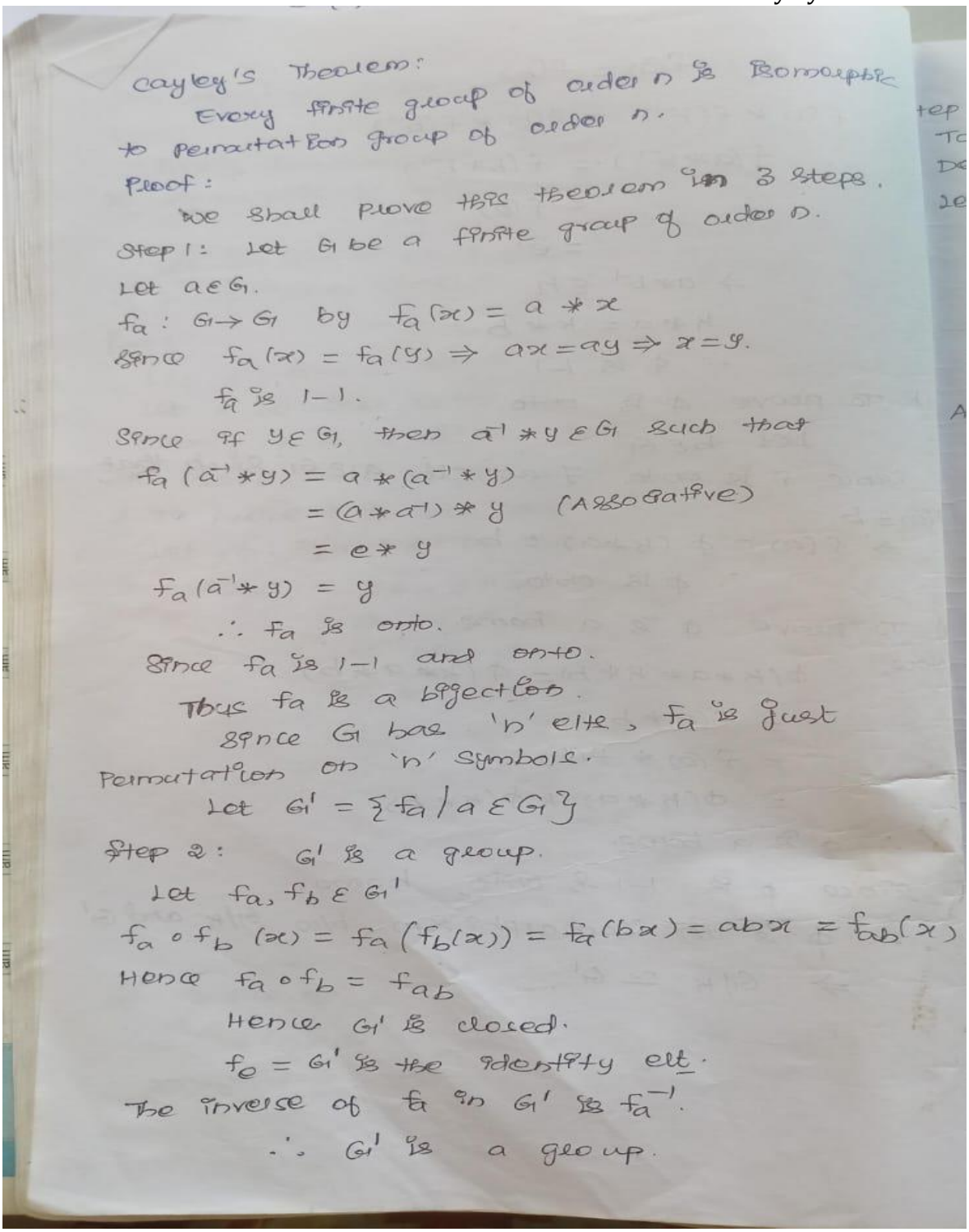


Cayley's Theorem:

Every finite group of order n is isomorphic to permutation group of order n.

Proof:

We shall prove this theorem in 3 steps.

Step 1: Let G be a finite group of order n.

Let $a \in G$.

$f_a : G \to G$ by $f_a(x) = a * x$

Since $f_a(x) = f_a(y) \Rightarrow ax = ay \Rightarrow x = y$.

f_a is 1–1.

Since if $y \in G$, then $a^{-1} * y \in G$ such that

$f_a(a^{-1} * y) = a * (a^{-1} * y)$

$= (a * a^{-1}) * y$ (Associative)

$= e * y$

$f_a(a^{-1} * y) = y$

$\therefore f_a$ is onto.

Since f_a is 1–1 and onto.

Thus f_a is a bijection.

Since G has 'n' elts, f_a is just permutation on 'n' symbols.

Let $G' = \{f_a / a \in G\}$

Step 2: G' is a group.

Let $f_a, f_b \in G'$

$f_a \circ f_b(x) = f_a(f_b(x)) = f_a(bx) = abx = f_{ab}(x)$

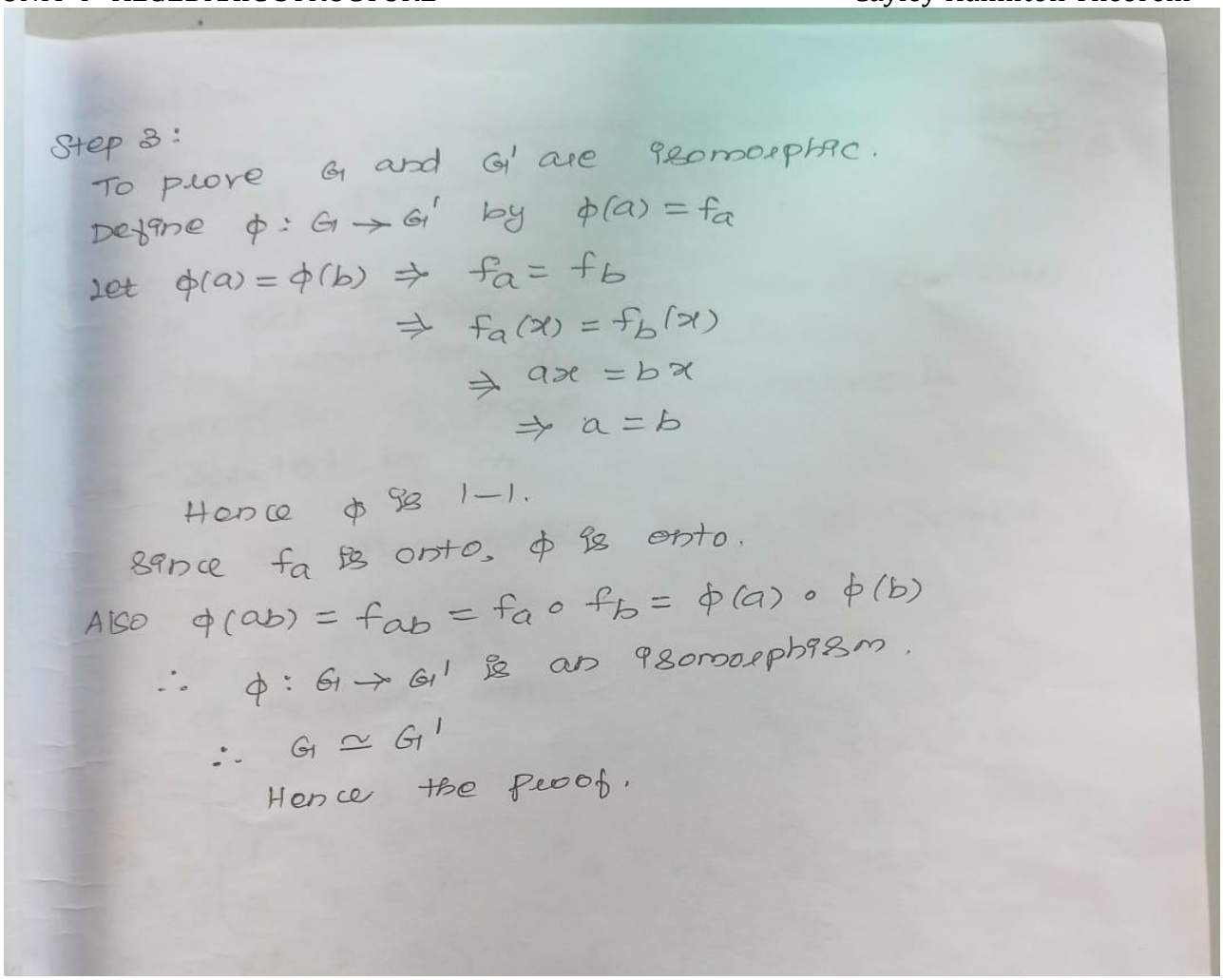
Hence $f_a \circ f_b = f_{ab}$

Hence G' is closed.

$f_e = G'$ is the identity elt.

The inverse of f_a in G' is f_a^{-1}.

$\therefore G'$ is a group.



Step 3:

To prove G and G' are isomorphic.

Define $\phi : G \to G'$ by $\phi(a) = f_a$

Let $\phi(a) = \phi(b) \Rightarrow f_a = f_b$

$\Rightarrow f_a(x) = f_b(x)$

$\Rightarrow ax = bx$

$\Rightarrow a = b$

Hence ϕ is 1–1.

Since f_a is onto, ϕ is onto.

Also $\phi(ab) = f_{ab} = f_a \circ f_b = \phi(a) \circ \phi(b)$

$\therefore \phi : G \to G'$ is an isomorphism.

$\therefore G \simeq G'$

Hence the proof.