Morphological Image Processing

Introduction

- **Morphology**: a branch of biology that deals with the form and structure of animals and plants

- Morphological image processing is used to extract image components for representation and description of region shape, such as boundaries, skeletons, and the convex hull

Preliminaries (1)

- **Reflection**

The reflection of a set B, denoted B, is defined as

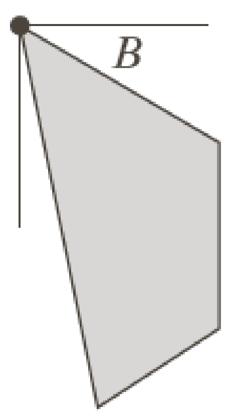
$$B = \{w \mid w = -b, \text{for } b \in B\}$$

- **Translation**

The translation of a set B by point $z = (z_1, z_2)$, denoted $(B)_Z$, is defined as

$$(B)_Z = \{c \mid c = b + z, \text{for } b \in B\}$$

Example: Reflection and Translation



a b c

FIGURE 9.1
(a) A set, (b) its reflection, and (c) its translation by z.

Preliminaries (2)

- **Structure elements (SE)**

Small sets or sub-images used to probe an image under study for properties of interest

Examples: Structuring Elements (1)

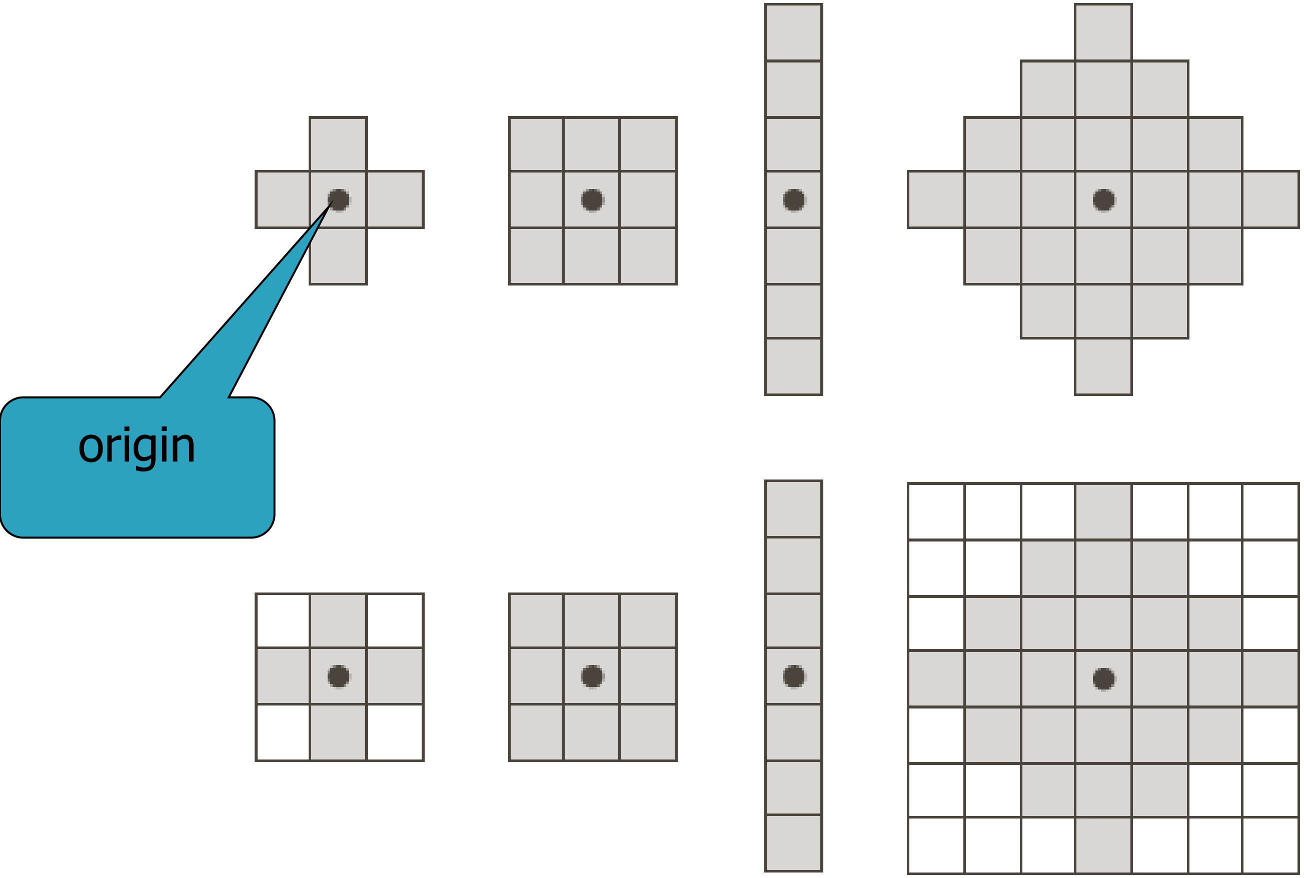
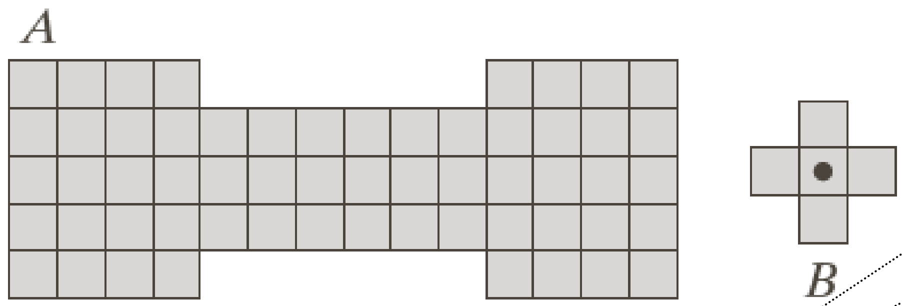


FIGURE 9.2 First row: Examples of structuring elements. Second row: Structuring elements converted to rectangular arrays. The dots denote the centers of the SEs.

Examples: Structuring Elements (2)

Accommodate the entire structuring elements when its origin is on the border of the original set A



Origin of B visits every element of A

At each location of the origin of B, if B is completely contained in A, then the location is a member of the new set, otherwise it is not a member of the new set.

a b
c d e

FIGURE 9.3 (a) A set (each shaded square is a member of the set). (b) A structuring element. (c) The set padded with background elements to form a rectangular array and provide a background border. (d) Structuring element as a rectangular array. (e) Set processed by the structuring element.

Erosion

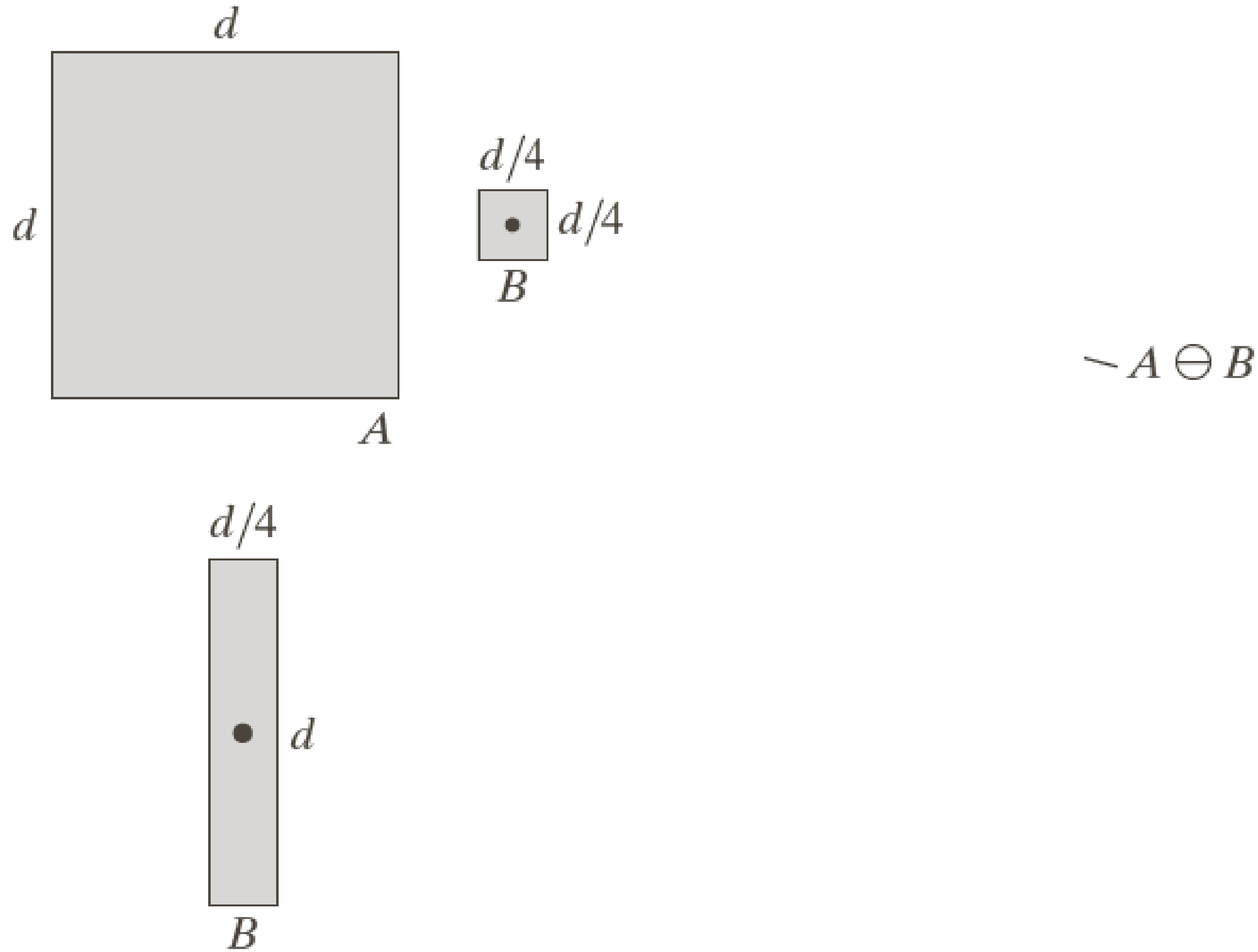
With A and B as sets in Z^2, the erosion of A by B, denoted $A \ominus B$, defined

$$A \ominus B = \left\{ z \mid (B)_Z \subseteq A \right\}$$

The set of all points z such that B, translated by z, is contained by A.

$$A \ominus B = \left\{ z \mid (B)_Z \cap A^c = \varnothing \right\}$$

Example of Erosion (1)



a b c
d e

FIGURE 9.4 (a) Set A. (b) Square structuring element, B. (c) Erosion of A by B, shown shaded. (d) Elongated structuring element. (e) Erosion of A by B using this element. The dotted border in (c) and (e) is the boundary of set A, shown only for reference.

Example of Erosion (2)

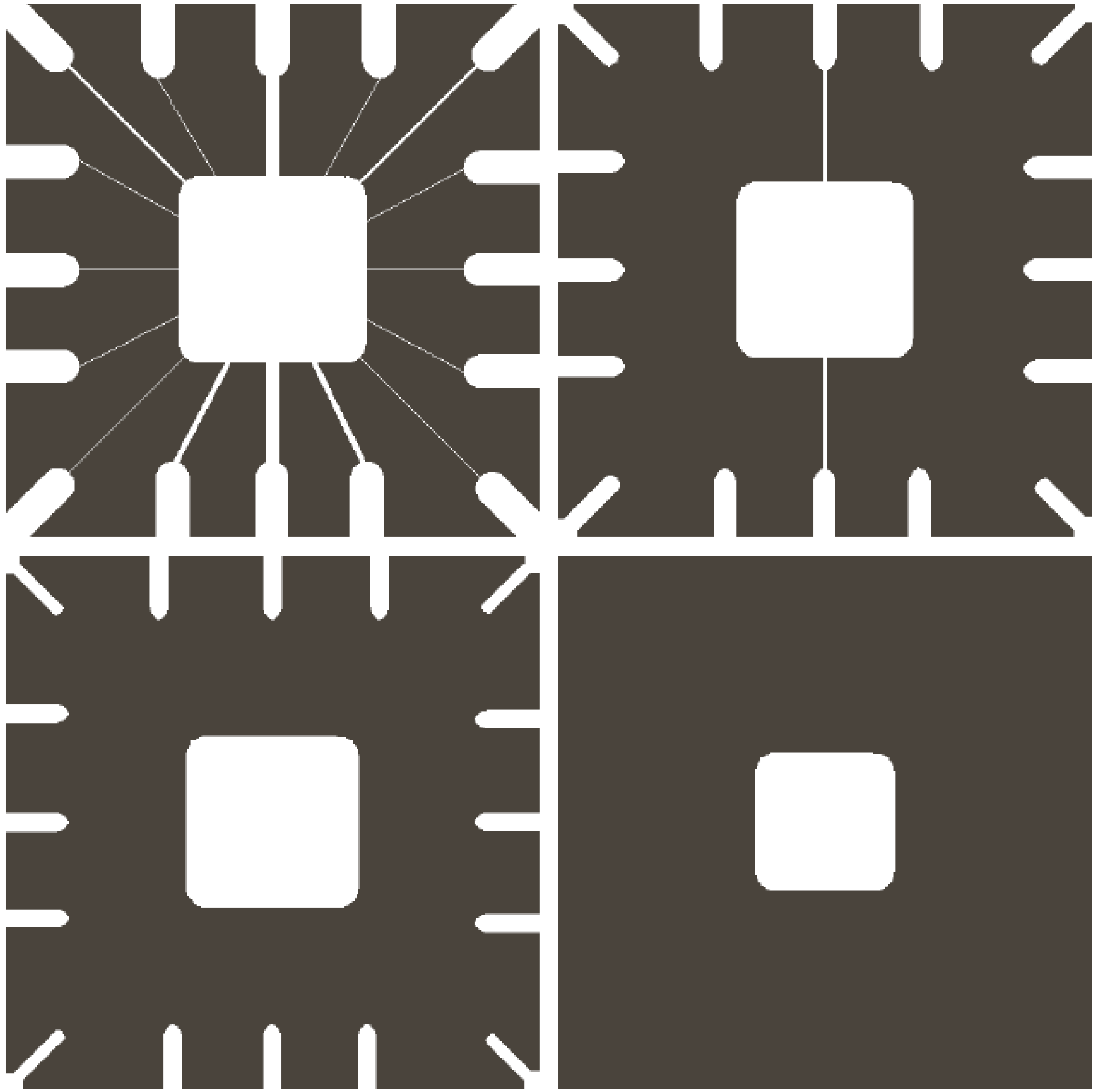


FIGURE 9.5 Using erosion to remove image components. (a) A 486 × 486 binary image of a wire-bond mask. (b)–(d) Image eroded using square structuring elements of sizes 11 × 11, 15 × 15, and 45 × 45, respectively. The elements of the SEs were all 1s.

Dilation

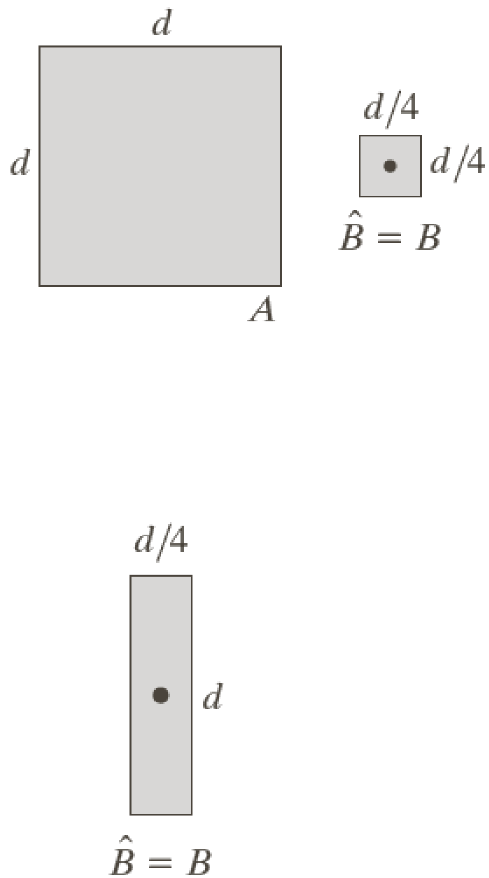
With A and B as sets in Z^2, the dilation of A by B, denoted $A \oplus B$, is defined as

$$\mathrm{A} \oplus \mathrm{B} = \left\{ z \mid \left(B \right)_z \cap A \neq \varnothing \right\}$$

The set of all displacements z, the translated B and A overlap by at least one element.

$$A \oplus B = \left\{ z \mid \left[\left(B \right)_z \cap A \right] \subseteq A \right\}$$

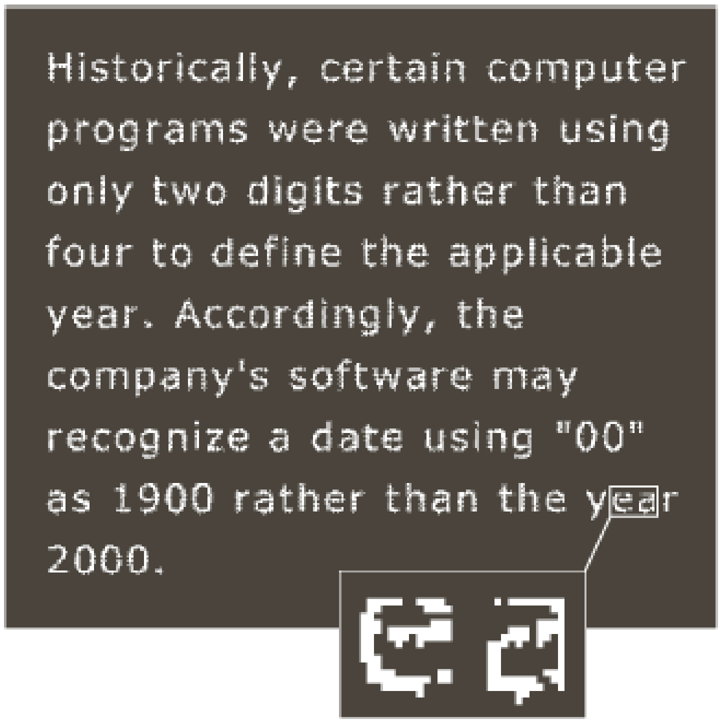
Examples of Dilation (1)



a	b	c
d		e

FIGURE 9.6
(a) Set A.
(b) Square structuring element (the dot denotes the origin).
(c) Dilation of A by B, shown shaded.
(d) Elongated structuring element. (e) Dilation of A using this element. The dotted border in (c) and (e) is the boundary of set A, shown only for reference

Examples of Dilation (2)



0	1	0
1	1	1
0	1	0

a c
b

FIGURE 9.7
(a) Sample text of poor resolution with broken characters (see magnified view).
(b) Structuring element.
(c) Dilation of (a) by (b). Broken segments were joined.

Duality

- Erosion and dilation are duals of each other with respect to set complementation and reflection

$$(A \ominus B)^c = A^c \oplus B$$

and

$$(A \oplus B)^c = A^c \ominus B$$

Duality

- Erosion and dilation are duals of each other with respect to set complementation and reflection

$$\left(A \ominus B\right)^c = \left\{z \mid \left(B\right)_Z \subseteq A\right\}^c$$

Duality

- Erosion and dilation are duals of each other with respect to set complementation and reflection

$$\left(A \oplus B\right)^{c} = \left\{ z \mid \left(B\right)_{Z} \cap A \neq \varnothing \right\}^{c}$$

Opening and Closing

- Opening generally smoothes the contour of an object, breaks narrow isthmuses, and eliminates thin protrusions

- Closing tends to smooth sections of contours but it generates fuses narrow breaks and long thin gulfs, eliminates small holes, and fills gaps in the contour

Opening and Closing

The opening of set A by structuring element B, denoted $A \circ B$, is defined as

$$A \circ B = \left(A \ominus B \right) \oplus B$$

The closing of set A by structuring element B, denoted $A \square B$, is defined as

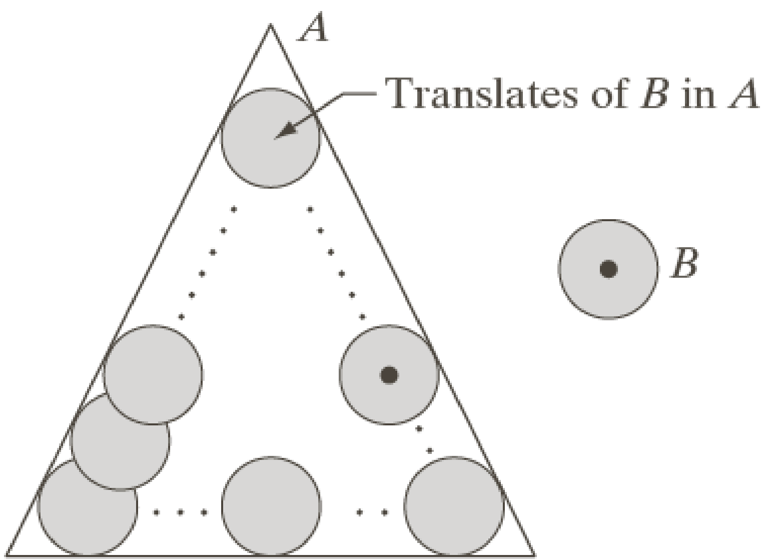
$$A \square B = \left(A \oplus B \right) \ominus B$$

Opening

The opening of set A by structuring element B, denoted $A \circ B$, is defined as

$$A \circ B = \bigcup \left\{ \left(B \right)_Z \mid \left(B \right)_Z \subseteq A \right\}$$

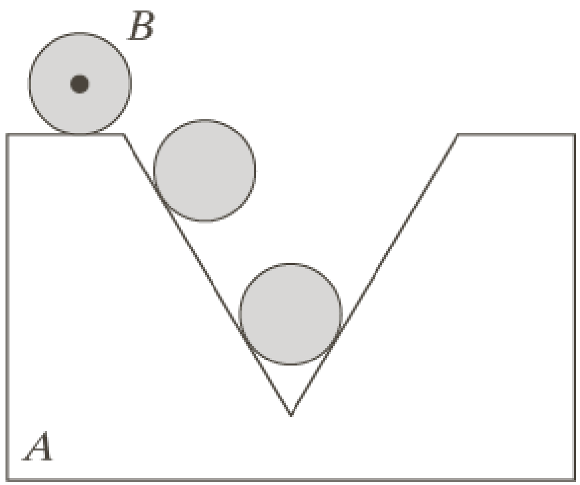
Example: Opening



a b c d

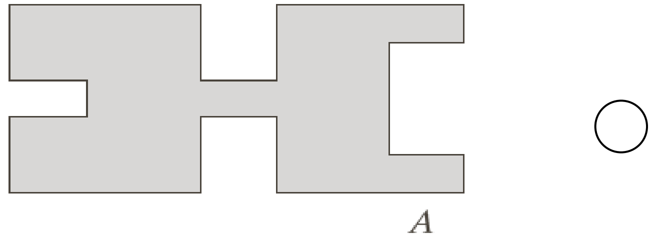
FIGURE 9.8 (a) Structuring element B "rolling" along the inner boundary of A (the dot indicates the origin of B). (b) Structuring element. (c) The heavy line is the outer boundary of the opening. (d) Complete opening (shaded). We did not shade A in (a) for clarity.

Example: Closing



a b c

FIGURE 9.9 (a) Structuring element B "rolling" on the outer boundary of set A. (b) The heavy line is the outer boundary of the closing. (c) Complete closing (shaded). We did not shade A in (a) for clarity.



a	
b	c
d	e
f	g
h	i

FIGURE 9.10 Morphological opening and closing. The structuring element is the small circle shown in various positions in (b). The SE was not shaded here for clarity. The dark dot is the center of the structuring element.

Duality of Opening and Closing

- Opening and closing are duals of each other with respect to set complementation and reflection

$$\left(A \bullet B\right)^{c} = (A^{c} \circ B)$$

$$\left(A \circ B\right)^{c} = (A^{c} \bullet B)$$

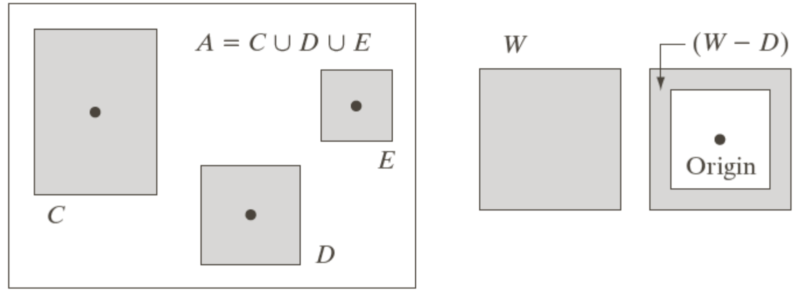
The Properties of Opening and Closing

- Properties of Opening

(a) $A \circ B$ is a subset (subimage) of A

- Properties of Closing

The Hit-or-Miss Transformation



a b
c d
e
f

FIGURE 9.12 (a) Set A. (b) A window, W, and the local background of D with respect to W, $(W - D)$. (c) Complement of A. (d) Erosion of A by D. (e) Erosion of A^c by $(W - D)$. (f) Intersection of (d) and (e), showing the location of the origin of D, as desired. The dots indicate the origins of C, D, and E.

if B denotes the set composed of D and its background,the match (or set of matches) of B in A, denoted $A \circledast B$,

$$A \circledast B = (A \ominus D) \cap \left[A^c \ominus (W - D) \right]$$

$$B = (B_1, B_2)$$

B_1 : object

B_2 : background

$$A \circledast B = (A \ominus B_1) \cap (A^c \ominus B_2)$$

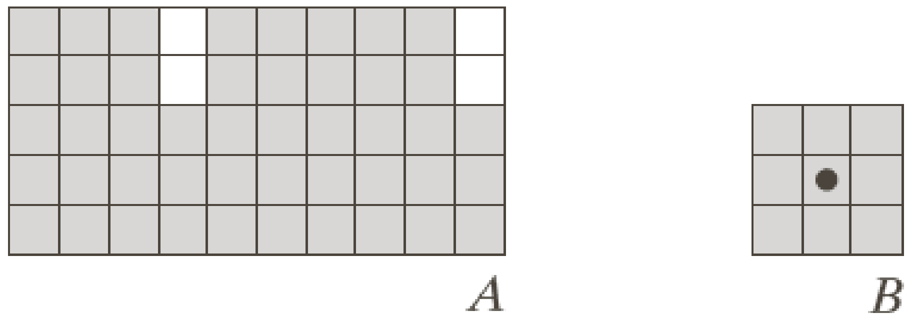
Some Basic Morphological Algorithms (1)

- **Boundary Extraction**

  The boundary of a set A, can be obtained by first eroding A by B and then performing the set difference between A and its erosion.

$$\beta(A) = A - \left(A \ominus B\right)$$

Example 1



$A \ominus B$ $\beta(A)$

a	b
c	d

FIGURE 9.13 (a) Set A. (b) Structuring element B. (c) A eroded by B. (d) Boundary, given by the set difference between A and its erosion.

Example 2

a b

FIGURE 9.14
(a) A simple binary image, with 1s represented in white. (b) Result of using Eq. (9.5-1) with the structuring element in Fig. 9.13(b).

Some Basic Morphological Algorithms (2)

- **Hole Filling**

  A hole may be defined as a background region surrounded by a connected border of foreground pixels.

  Let A denote a set whose elements are 8-connected boundaries, each boundary enclosing a background region (i.e., a hole). Given a point in each hole, the objective is to fill all the holes with 1s.

Some Basic Morphological Algorithms (2)

- **Hole Filling**

  1. Forming an array X_0 of 0s (the same size as the array containing A), except the locations in X_0 corresponding to the given point in each hole, which we set to 1.

  2. $X_k = (X_{k-1} \oplus B) \cap A^c \quad k=1,2,3,\ldots$

  Stop the iteration if $X_k = X_{k-1}$

Exan

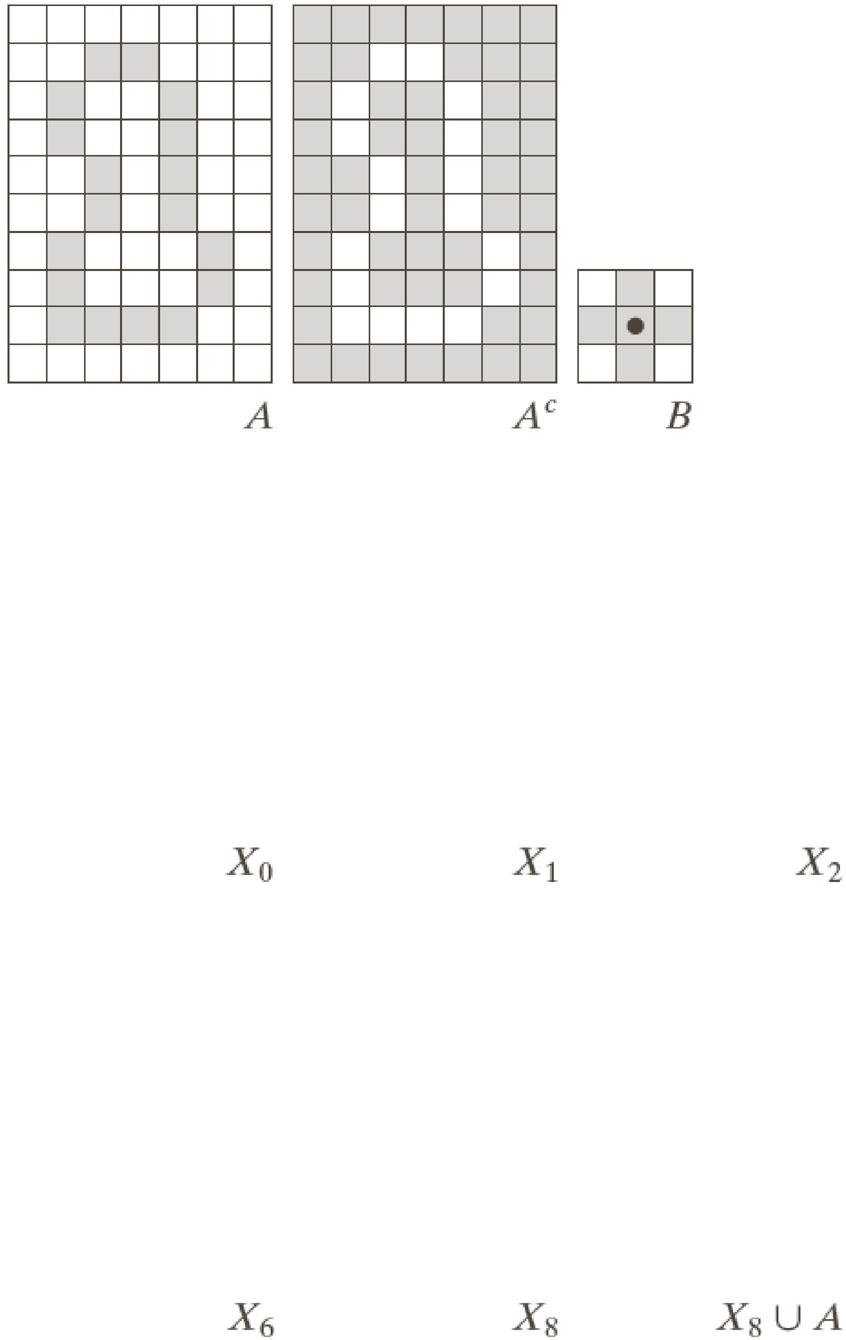
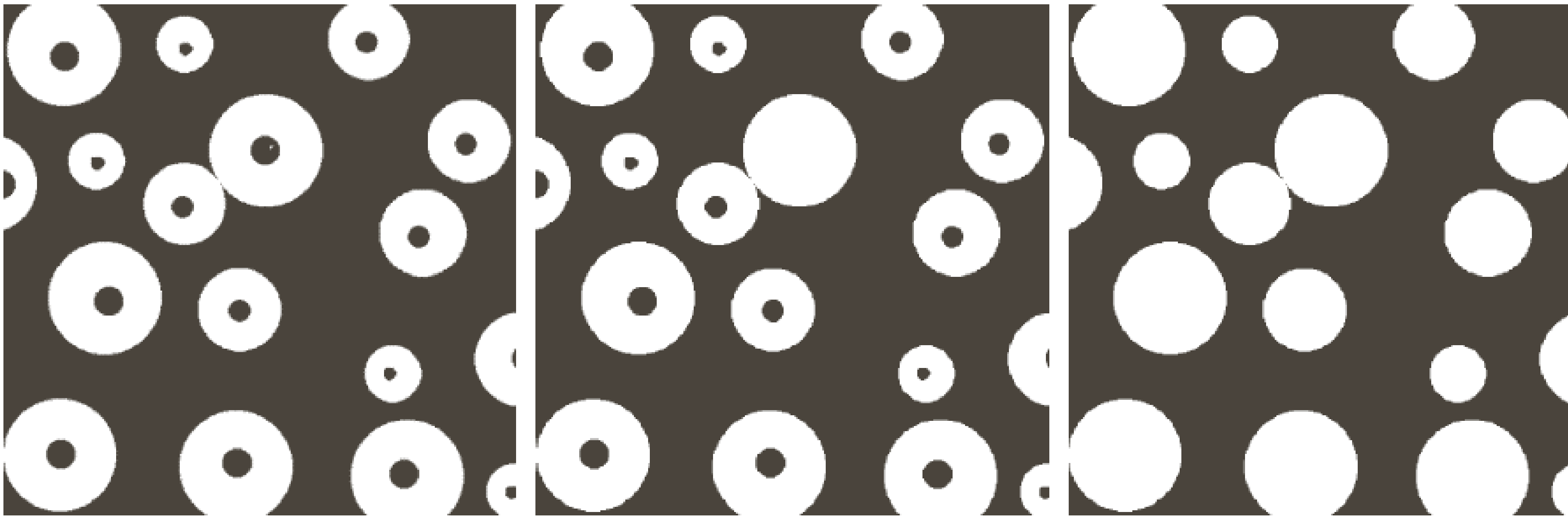


FIGURE 9.15 Hole filling. (a) Set A (shown shaded). (b) Complement of A. (c) Structuring element B. (d) Initial point inside the boundary. (e)–(h) Various steps of Eq. (9.5-2). (i) Final result [union of (a) and (h)].



a b c

FIGURE 9.16 (a) Binary image (the white dot inside one of the regions is the starting point for the hole-filling algorithm). (b) Result of filling that region. (c) Result of filling all holes.

Some Basic Morphological Algorithms (3)

- **Extraction of Connected Components**

  Central to many automated image analysis applications.

  Let A be a set containing one or more connected components, and form an array X_0 (of the same size as the array containing A) whose elements are 0s, except at each location known to correspond to a point in each connected component in A, which is set to 1.

Some Basic Morphological Algorithms (3)

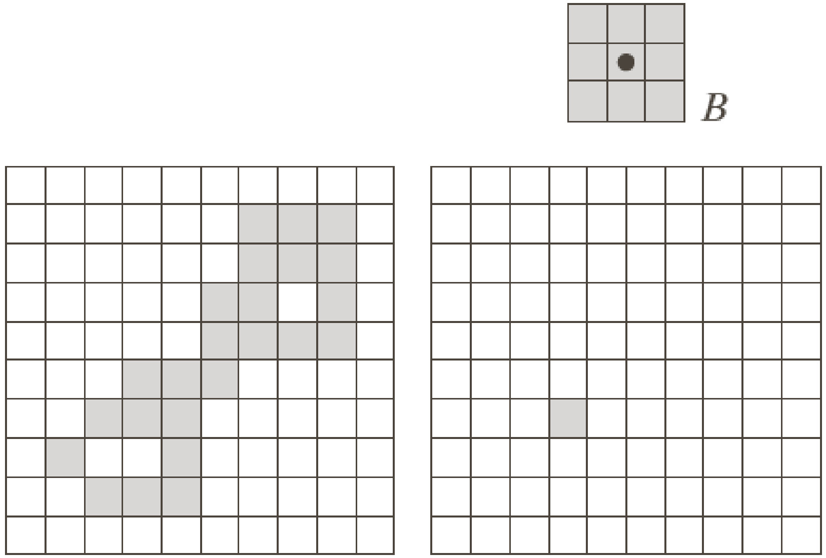
- **Extraction of Connected Components**

  Central to many automated image analysis applications.

$$X_k = (X_{k-1} \oplus B) \cap A$$

$$B : \text{structuring element}$$

$$\text{until } X_k = X_{k\text{-}1}$$

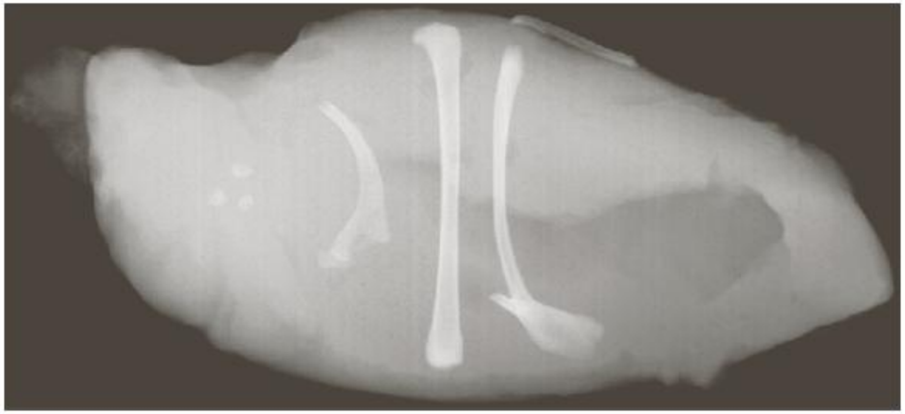


X_1

X_2 X_3 X_6

	a	
b	c	d
e	f	g

FIGURE 9.17 Extracting connected components. (a) Structuring element. (b) Array containing a set with one connected component. (c) Initial array containing a 1 in the region of the connected component. (d)–(g) Various steps in the iteration of Eq. (9.5-3).



Connected component	No. of pixels in connected comp
01	11
02	9
03	9
04	39
05	133
06	1
07	1
08	743
09	7
10	11
11	11
12	9
13	9
14	674
15	85

a
b
c d

FIGURE 9.18 (a) X-ray image of chicken filet with bone fragments. (b) Thresholded image. (c) Image eroded with a 5 × 5 structuring element of 1s. (d) Number of pixels in the connected components of (c). (Image courtesy of NTB Elektronische Geraete GmbH, Diepholz, Germany, www.ntbxray.com.)

Some Basic Morphological Algorithms (4)

- **Convex Hull**

  A set A is said to be ***convex*** if the straight line segment joining any two points in A lies entirely within A.

  The ***convex hull*** H or of an arbitrary set S is the smallest convex set containing S.

Some Basic Morphological Algorithms (4)

- **Convex Hull**

Let B^i, $i = 1,\ 2,\ 3,\ 4$, represent the four structuring elements. The procedure consists of implementing the equation:

$$X_k^i = (X_{k-1} \circledast B^i) \cup A$$
$$i = 1, 2, 3, 4 \ \text{ and } \ k = 1, 2, 3, \ldots$$

with $X_0^i = A.$

When the procedure converges, or $X_k^i = X_{k-1}^i$, let $D^i = X_k^i$, the convex hull of A is

$$C(A) = \bigcup_{i=1}^{4} D^i$$

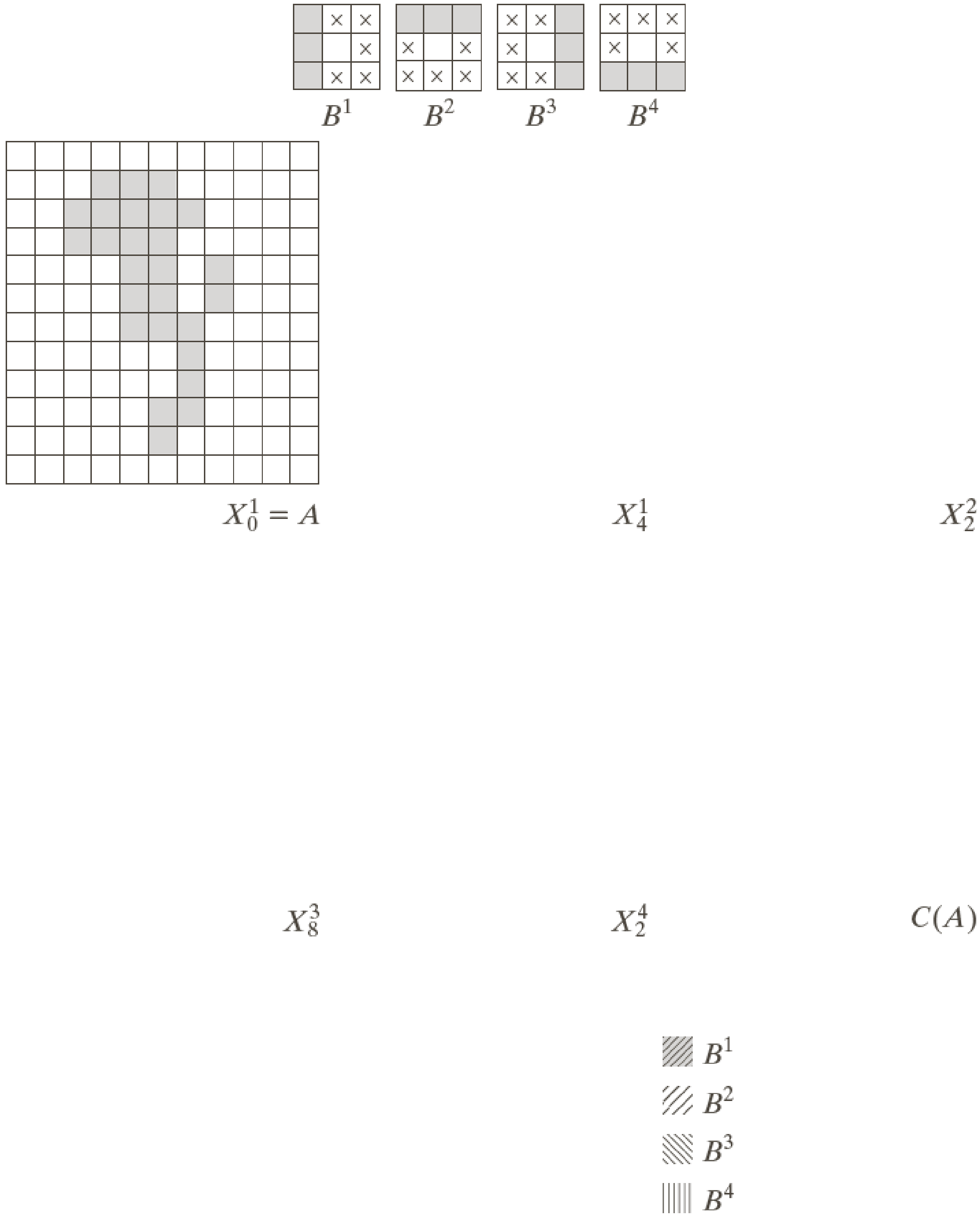


FIGURE 9.19
(a) Structuring elements. (b) Set A. (c)–(f) Results of convergence with the structuring elements shown in (a). (g) Convex hull. (h) Convex hull showing the contribution of each structuring element.

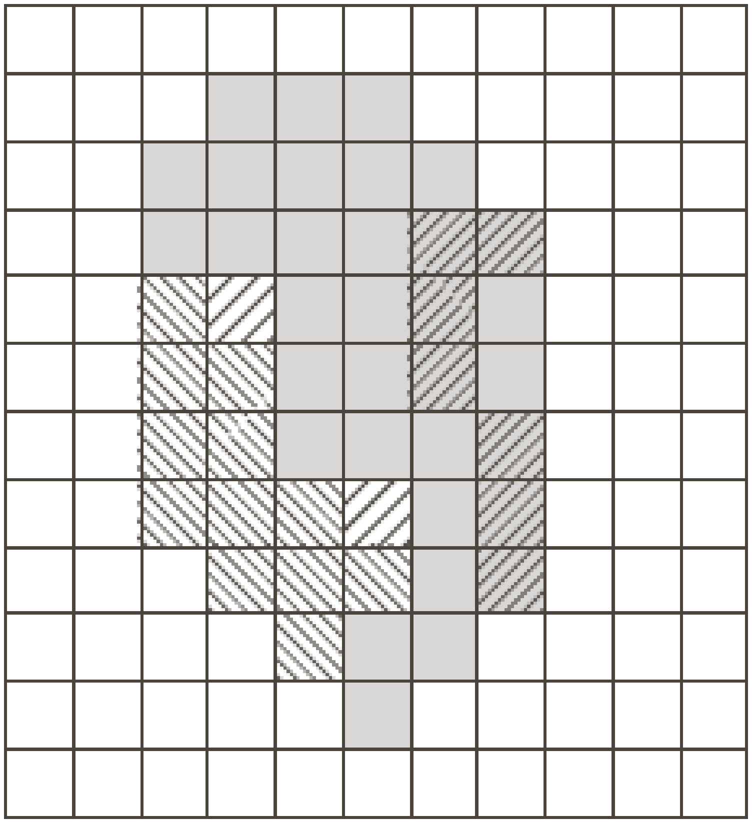


FIGURE 9.20 Result of limiting growth of the convex hull algorithm to the maximum dimensions of the original set of points along the vertical and horizontal directions.

Some Basic Morphological Algorithms (5)

- Thinning

 The thinning of a set A by a structuring element B, defined

$$A \otimes B = A - (A \circledast B)$$

$$= A \cap (A \circledast B)^c$$

Some Basic Morphological Algorithms (5)

- A more useful expression for thinning A symmetrically is based on a sequence of structuring elements:

$$\{B\} = \{B^1, B^2, B^3, ..., B^n\}$$

where B^i is a rotated version of $B^{i\text{-}1}$

The thinning of A by a sequence of structuring element $\{B\}$

$$A \otimes \{B\} = ((...((A \otimes B^1) \otimes B^2)...) \otimes B^n)$$

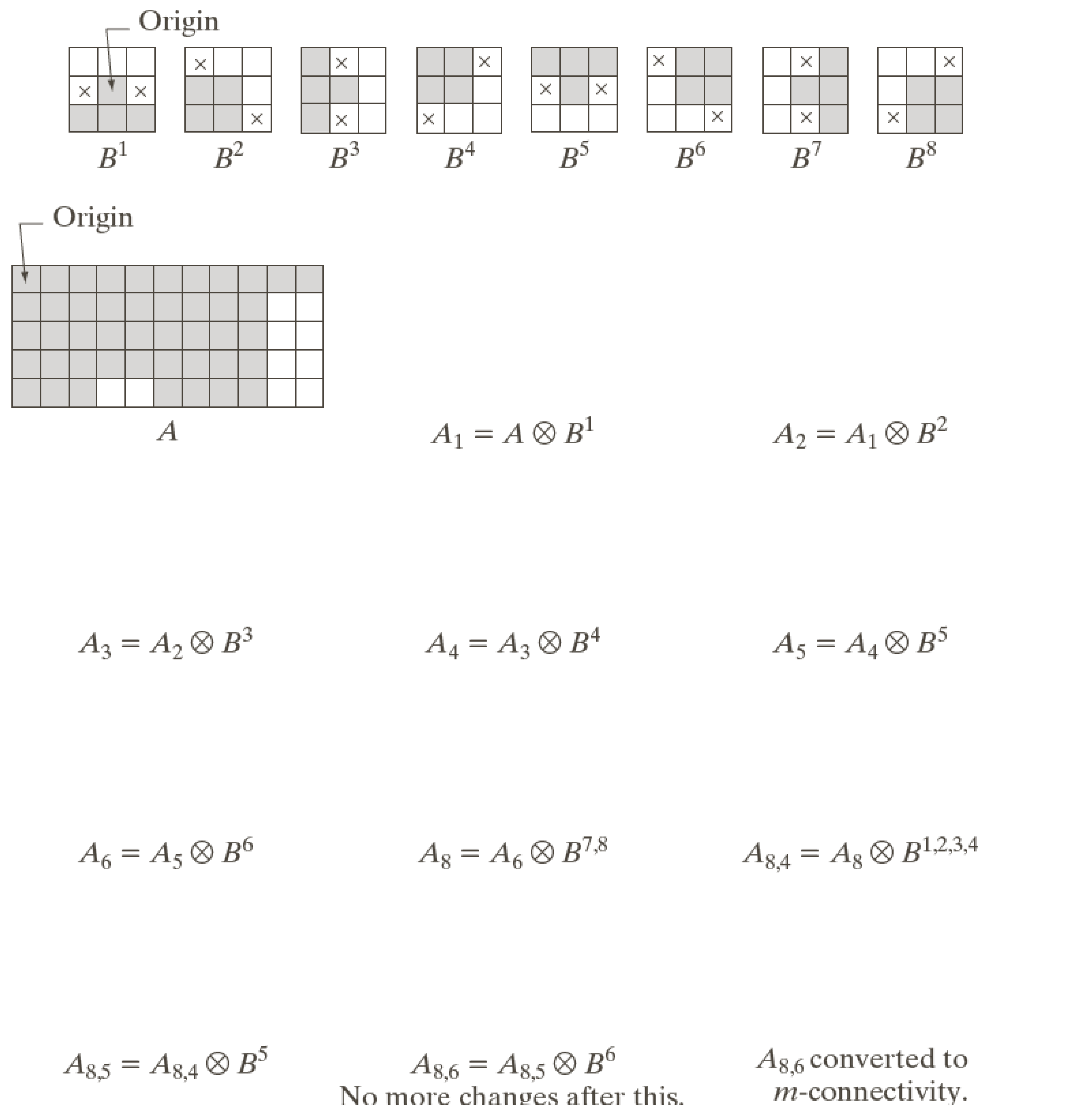


FIGURE 9.21 (a) Sequence of rotated structuring elements used for thinning. (b) Set A. (c) Result of thinning with the first element. (d)–(i) Results of thinning with the next seven elements (there was no change between the seventh and eighth elements). (j) Result of using the first four elements again. (l) Result after convergence. (m) Conversion to m-connectivity.

Some Basic Morphological Algorithms (6)

- Thickening:

The thickening is defined by the expression

$$A \odot B = A \cup \left(A \circledast B \right)$$

The thickening of A by a sequence of structuring element $\{B\}$

$$A \odot \{B\} = ((\ldots((A \odot B^1) \odot B^2)\ldots) \odot B^n)$$

In practice, the usual procedure is to thin the background of the set and then complement the result.

Some Basic Morphological Algorithms (6)

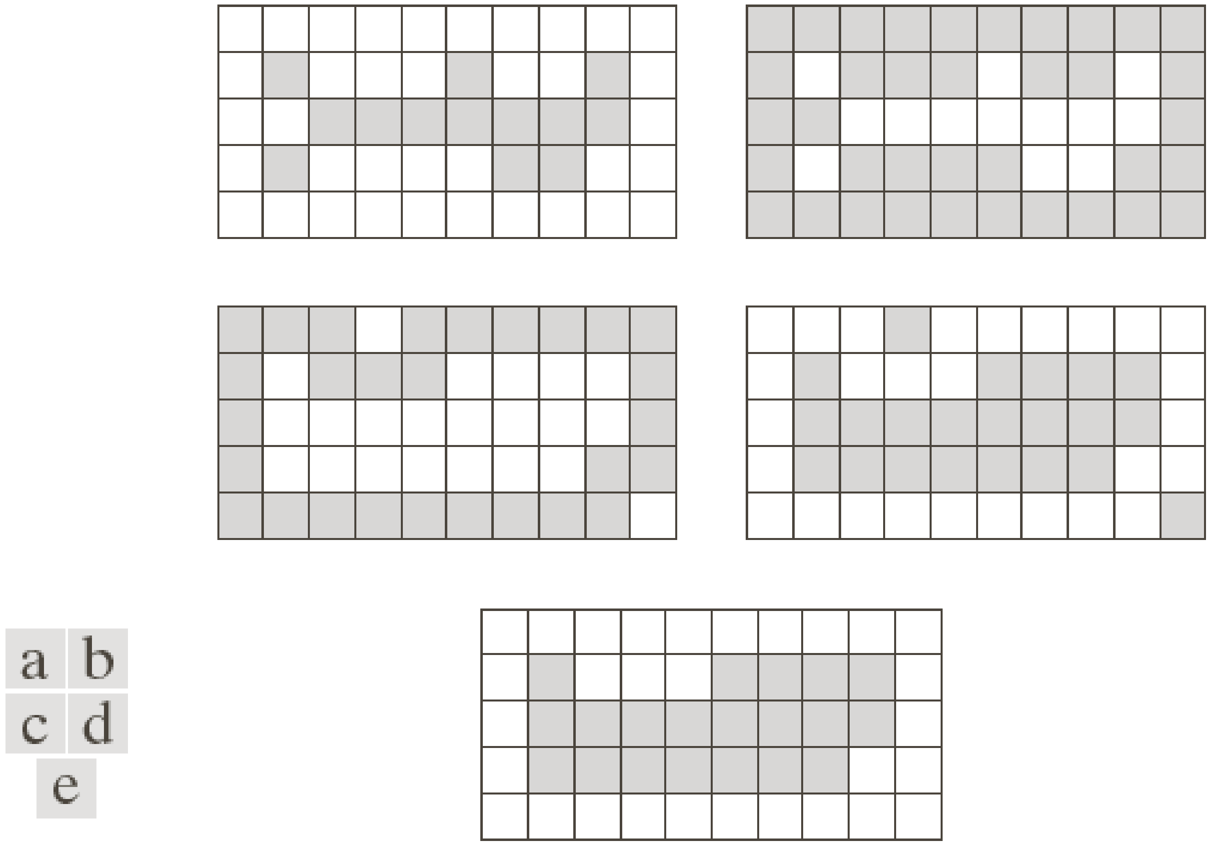


FIGURE 9.22 (a) Set A. (b) Complement of A. (c) Result of thinning the complement of A. (d) Thickened set obtained by complementing (c). (e) Final result, with no disconnected points.

Some Basic Morphological Algorithms (7)

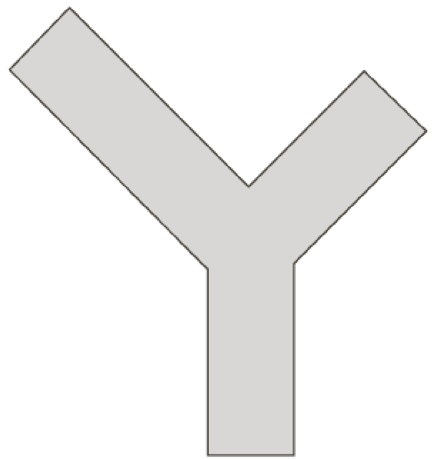
- Skeletons

A skeleton, $S(A)$ of a set A has the following properties

a. if z is a point of $S(A)$ and $(D)_z$ is the largest disk centered at z and contained in A, one cannot find a larger disk containing $(D)_z$ and included in A. The disk $(D)_z$ is called a maximum disk.

b. The disk $(D)_z$ touches the boundary of A at two or more different places.

Some Basic Morphological Algorithms (7)



a	b
c	d

FIGURE 9.23
(a) Set A.
(b) Various positions of maximum disks with centers on the skeleton of A.
(c) Another maximum disk on a different segment of the skeleton of A.
(d) Complete skeleton.

Some Basic Morphological Algorithms (7)

The skeleton of A can be expressed in terms of erosion and openings.

$$S(A) = \bigcup_{k=0}^{K} S_k(A)$$

with $K = \max\{k \mid A \ominus kB \neq \phi\}$;

$$S_k(A) = (A \ominus kB) - (A \ominus kB) \circ B$$

where B is a structuring element, and

$$(A \ominus kB) = ((..((A \ominus B) \ominus B) \ominus ...) \ominus B)$$

k successive erosions of A.

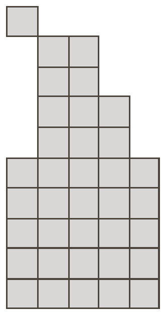
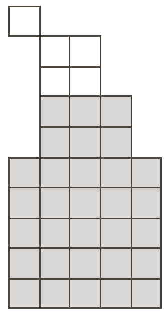
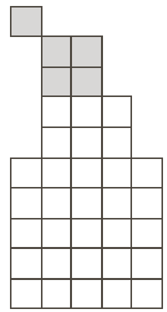
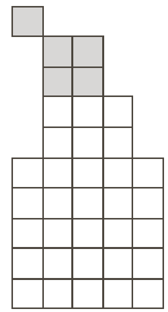
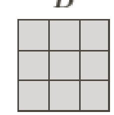
k	$A \ominus kB$	$(A \ominus kB) \circ B$	$S_k(A)$	$\bigcup_{k=0}^{K} S_k(A)$
0				

FIGURE 9.24 Implementation of Eqs. (9.5-11) through (9.5-15). The original set is at the top left, and its morphological skeleton is at the bottom of the fourth column. The reconstructed set is at the bottom of the sixth column.

B



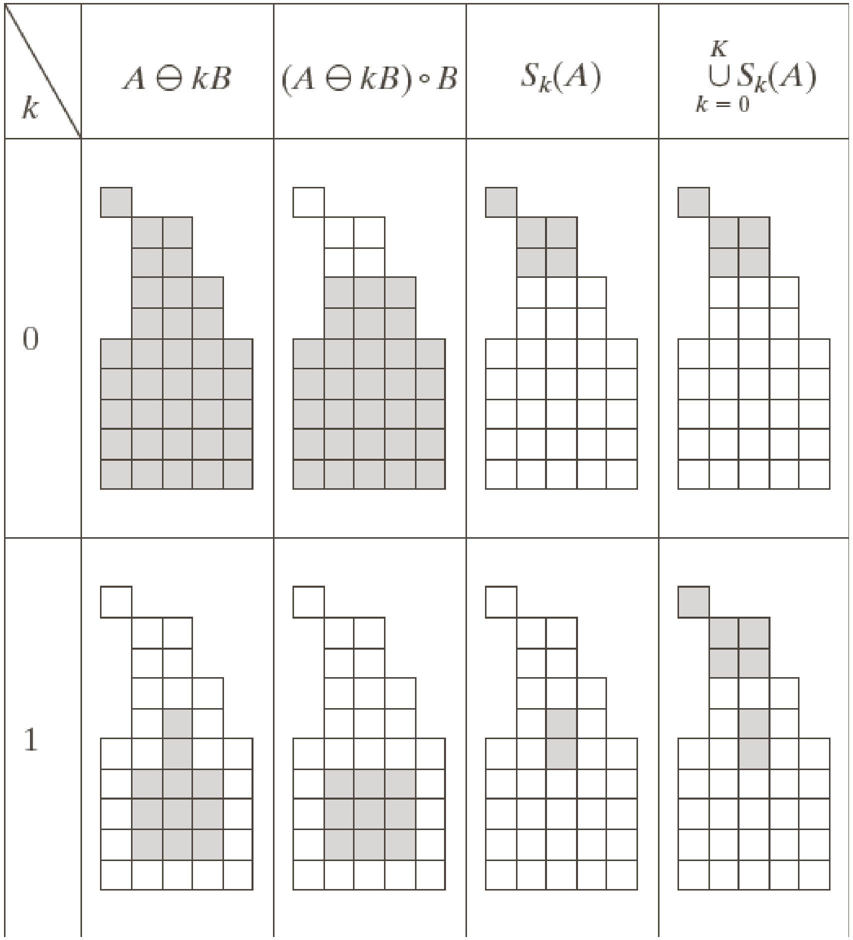
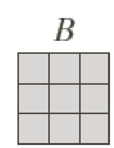


FIGURE 9.24 Implementation of Eqs. (9.5-11) through (9.5-15). The original set is at the top left, and its morphological skeleton is at the bottom of the fourth column. The reconstructed set is at the bottom of the sixth column.



Some Basic Morphological Algorithms (7)

A can be reconstructed from the subsets by using

$$A = \bigcup_{k=0}^{K} (S_k(A) \oplus kB)$$

where $S_k(A) \oplus kB$ denotes k successive dilations of A.

$$(S_k(A) \oplus kB) = ((\ldots((S_k(A) \oplus B) \oplus B)\ldots \oplus B)$$

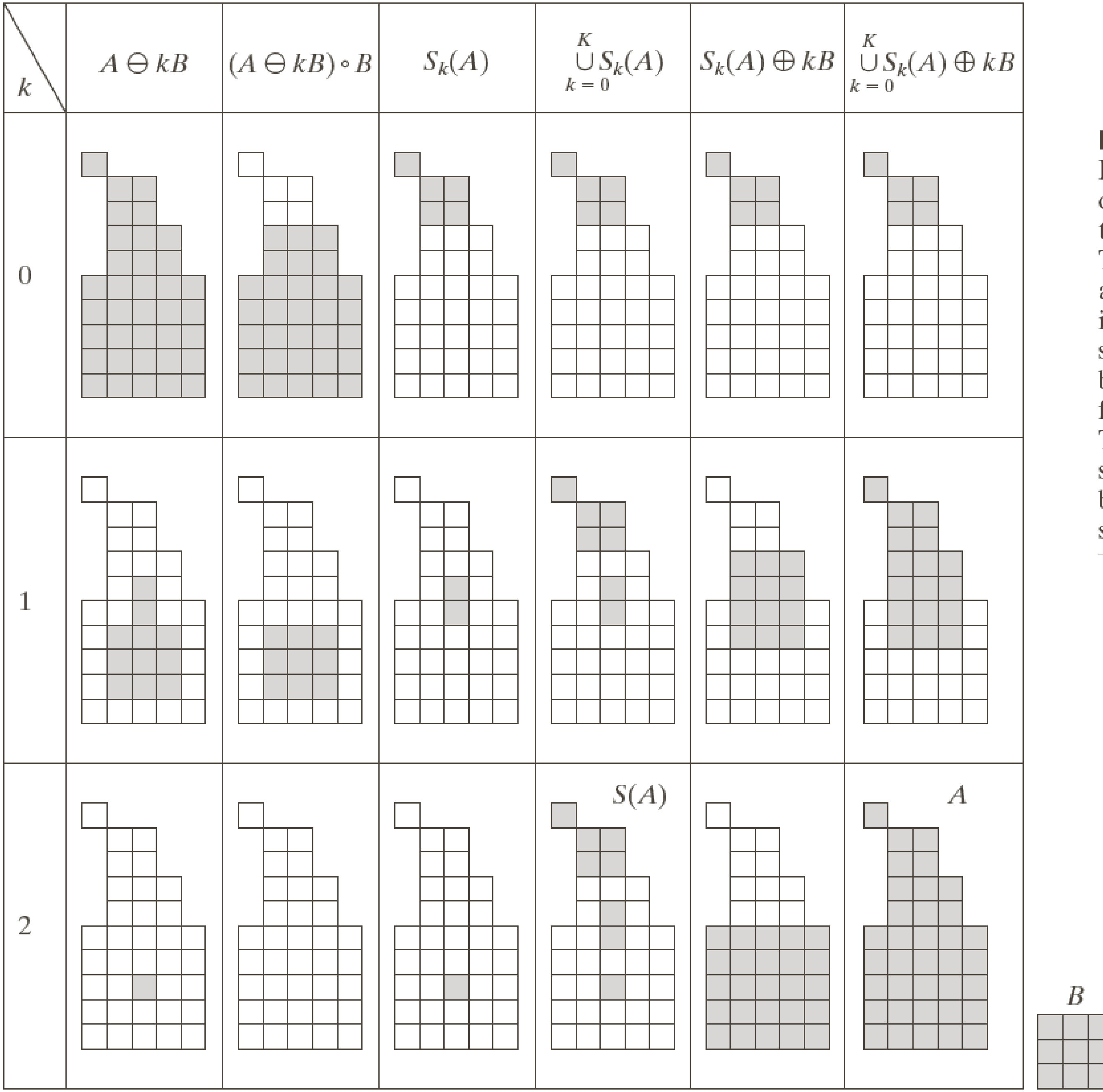


FIGURE 9.24 Implementation of Eqs. (9.5-11) through (9.5-15). The original set is at the top left, and its morphological skeleton is at the bottom of the fourth column. The reconstructed set is at the bottom of the sixth column.

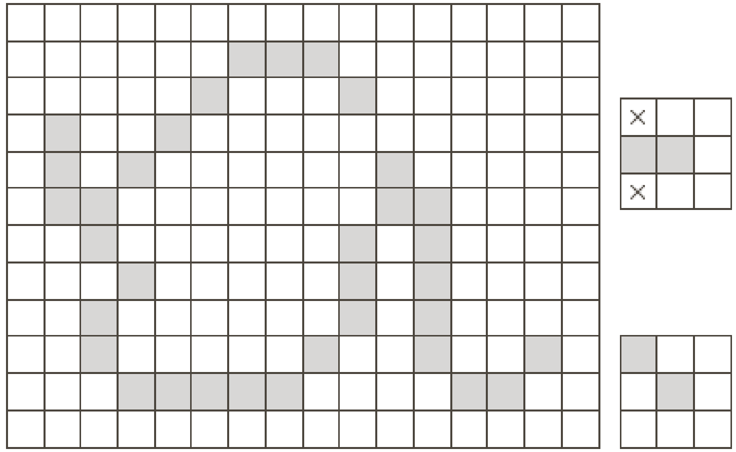
Some Basic Morphological Algorithms (8)

- Pruning

a. Thinning and skeletonizing tend to leave parasitic components

b. Pruning methods are essential complement to thinning and skeletonizing procedures

Pruning: Example



B^1, B^2, B^3, B^4 (rotated 90°)

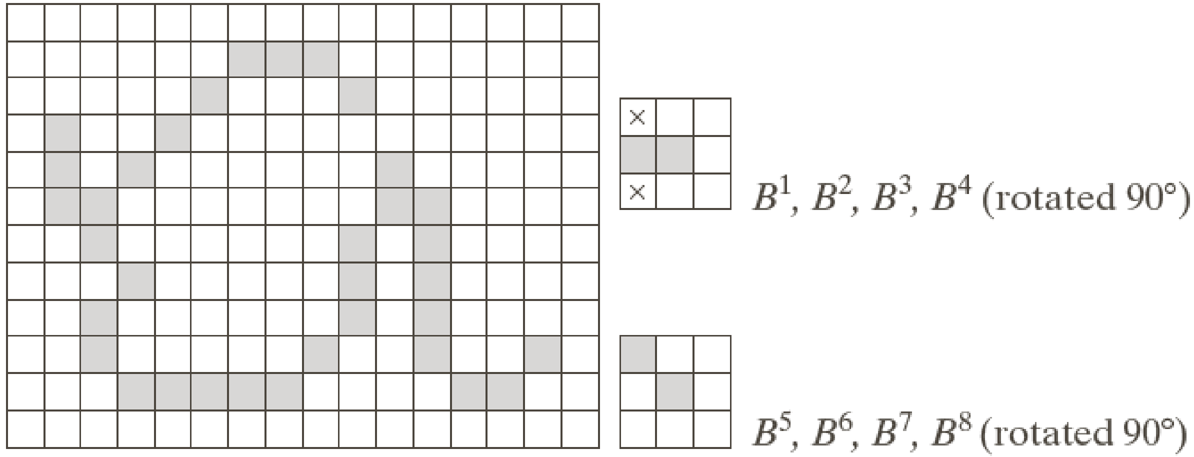
B^5, B^6, B^7, B^8 (rotated 90°)

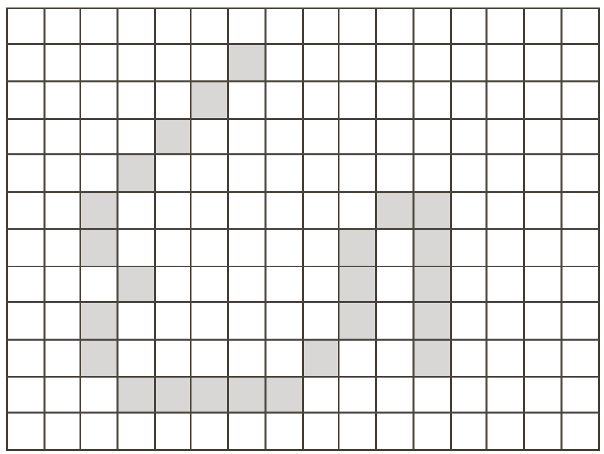
$$X_1 = A \otimes \{B\}$$

a b
c
d e
f g

FIGURE 9.25 (a) Original image. (b) and (c) Structuring elements used for deleting end points. (d) Result of three cycles of thinning. (e) End points of (d). (f) Dilation of end points conditioned on (a). (g) Pruned image.

Pruning: Example



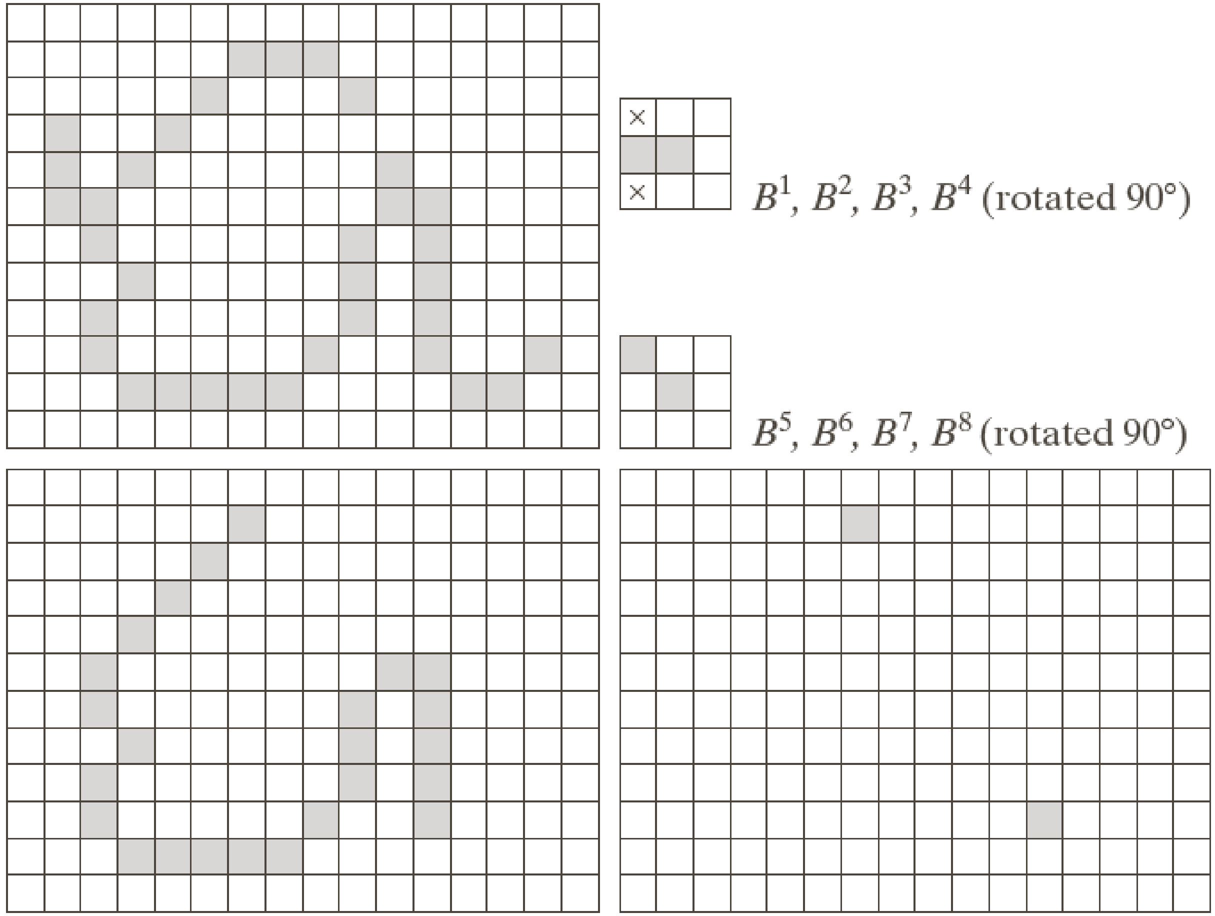


$$X_2 = \bigcup_{k=1}^{8} \left(X_1 \circledast B^k \right)$$

a b
c
d e
f g

FIGURE 9.25 (a) Original image. (b) and (c) Structuring elements used for deleting end points. (d) Result of three cycles of thinning. (e) End points of (d). (f) Dilation of end points conditioned on (a). (g) Pruned image.

Pruning: Example



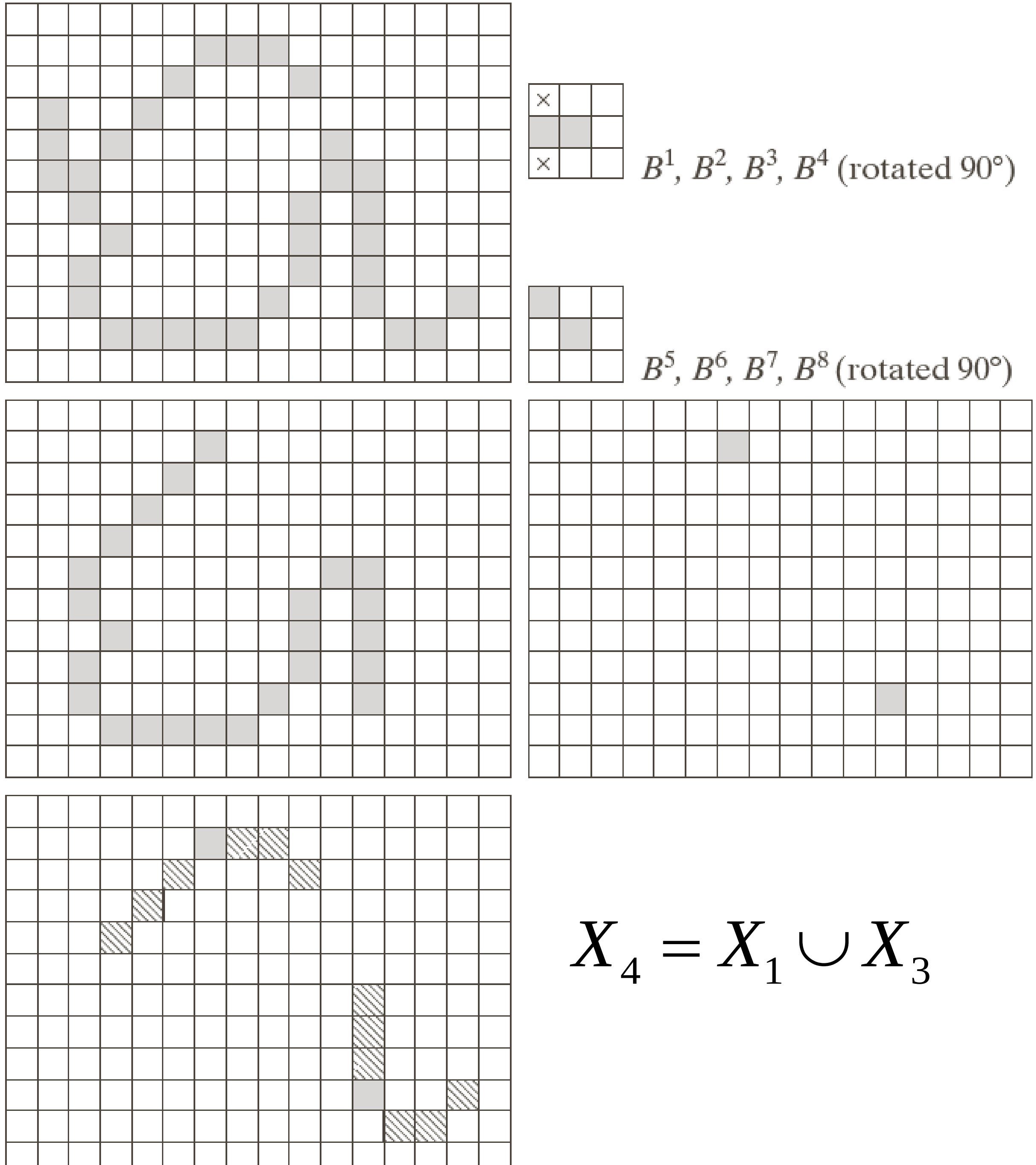
a b
c
d e
f g

FIGURE 9.25 (a) Original image. (b) and (c) Structuring elements used for deleting end points. (d) Result of three cycles of thinning. (e) End points of (d). (f) Dilation of end points conditioned on (a). (g) Pruned image.

$$X_3 = \left(X_2 \oplus H\right) \cap A$$

$H:3\times 3$ structuring element

Pruning: Example

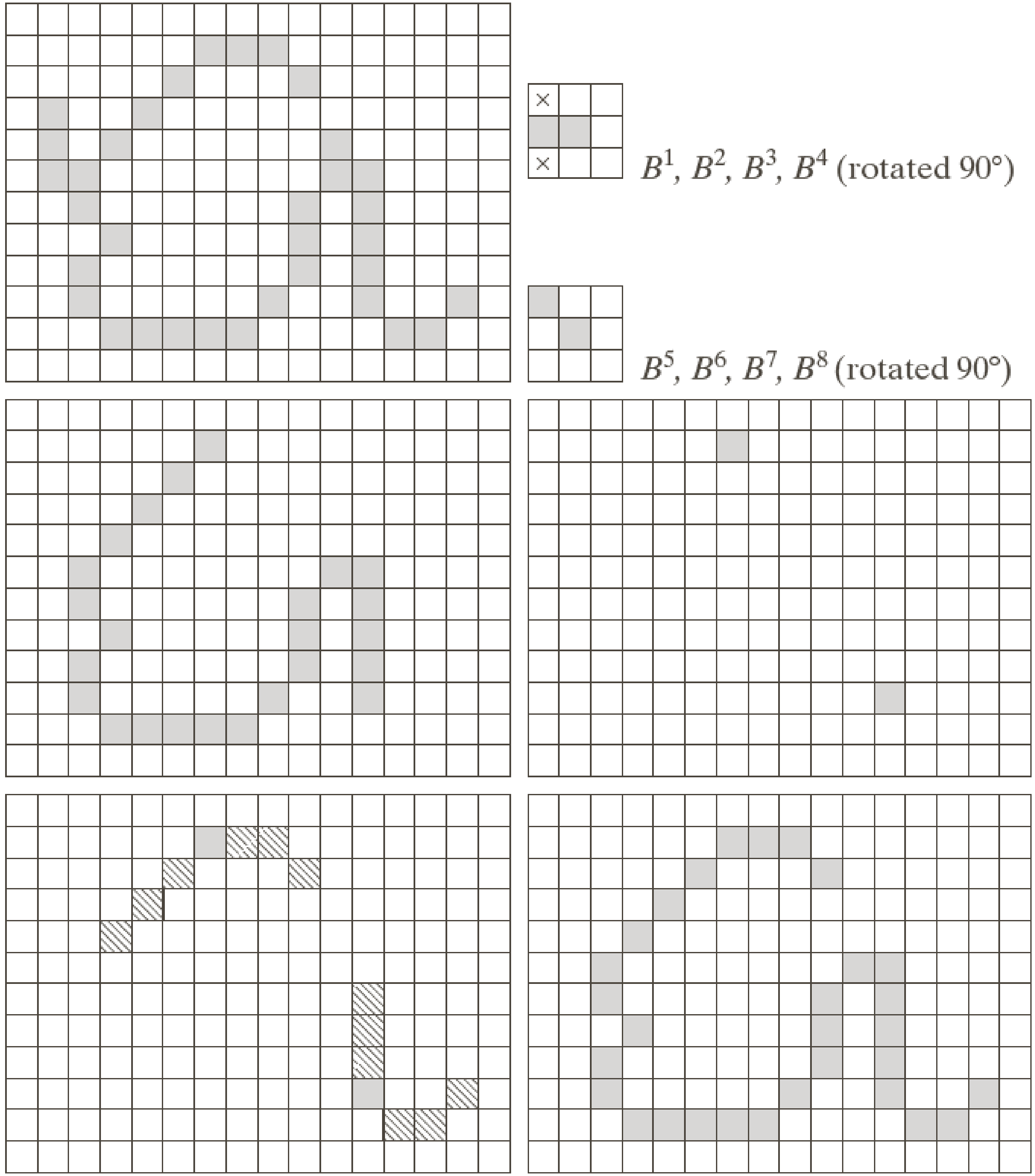


a b
c
d e
f g

FIGURE 9.25 (a) Original image. (b) and (c) Structuring elements used for deleting end points. (d) Result of three cycles of thinning. (e) End points of (d). (f) Dilation of end points conditioned on (a). (g) Pruned image.

$$X_4 = X_1 \cup X_3$$

Pruning: Example



a b
c
d e
f g

FIGURE 9.25 (a) Original image. (b) and (c) Structuring elements used for deleting end points. (d) Result of three cycles of thinning. (e) End points of (d). (f) Dilation of end points conditioned on (a). (g) Pruned image.

Some Basic Morphological Algorithms (9)

- Morphological Reconstruction

It involves two images and a structuring element

a. One image contains the starting points for the transformation (The image is called marker)
b. Another image (mask) constrains the transformation
c. The structuring element is used to define connectivity

Morphological Reconstruction: Geodesic Dilation

Let F denote the marker image and G the mask image, $F \subseteq G$. The geodesic dilation of size 1 of the marker image with respect to the mask, denoted by $D_G^{(1)}(F)$, is defined as

$$D_G^{(1)}(F) = \left(F \oplus B\right) \cap \mathrm{G}$$

The geodesic dilation of size n of the marker image F with respect to G, denoted by $D_G^{(n)}(F)$, is defined as

$$D_G^{(n)}(F) = D_G^{(1)}(F)\left[D_G^{(n-1)}(F)\right]$$

with $D_G^{(0)}(F) = F$.

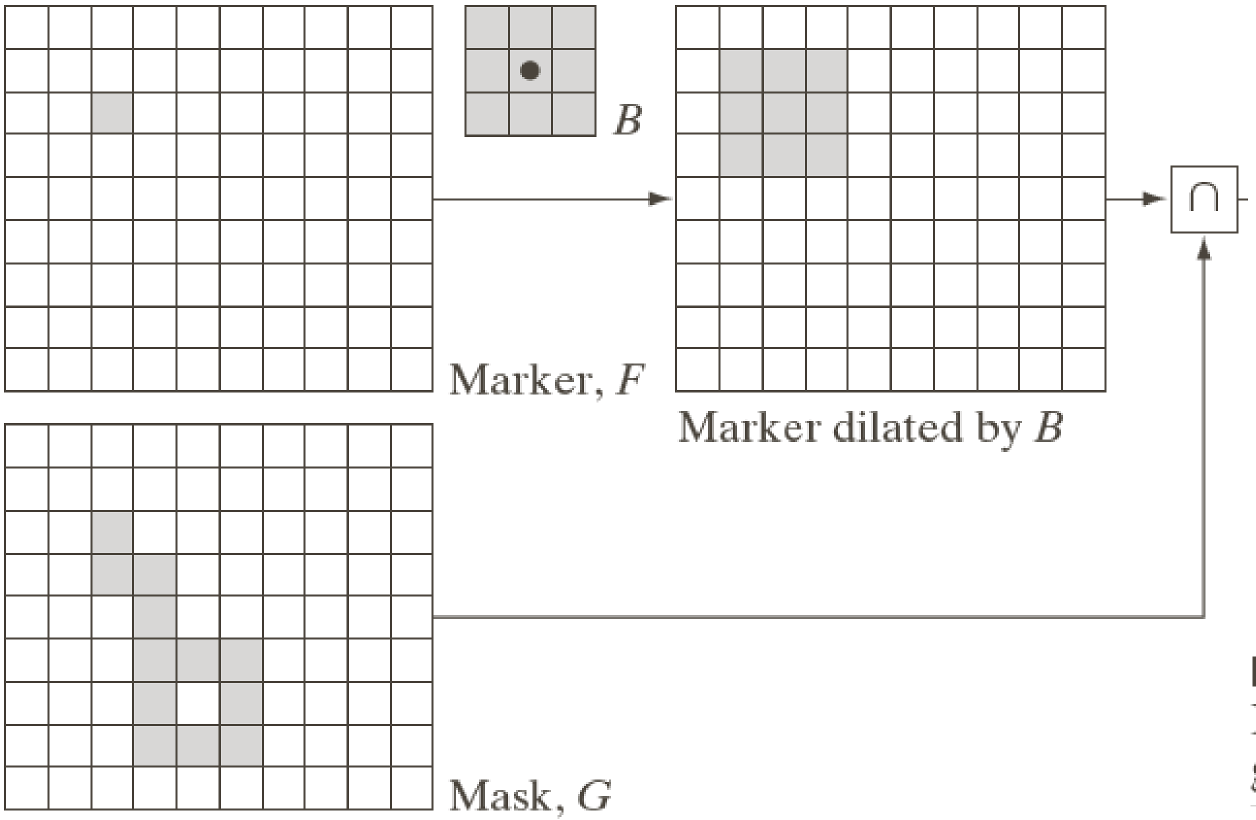


FIGURE 9.26 Illustration of geodesic dilation.

Morphological Reconstruction: Geodesic Erosion

Let F denote the marker image and G the mask image. The geodesic erosion of size 1 of the marker image with respect to the mask, denoted by $E_G^{(1)}(F)$, is defined as

$$E_G^{(1)}(F) = \left(F \ominus B\right) \cup \mathrm{G}$$

The geodesic erosion of size n of the marker image F with respect to G, denoted by $E_G^{(n)}(F)$, is defined as

$$E_G^{(n)}(F) = E_G^{(1)}(F)\left[E_G^{(n-1)}(F)\right]$$

with $E_G^{(0)}(F) = F$.

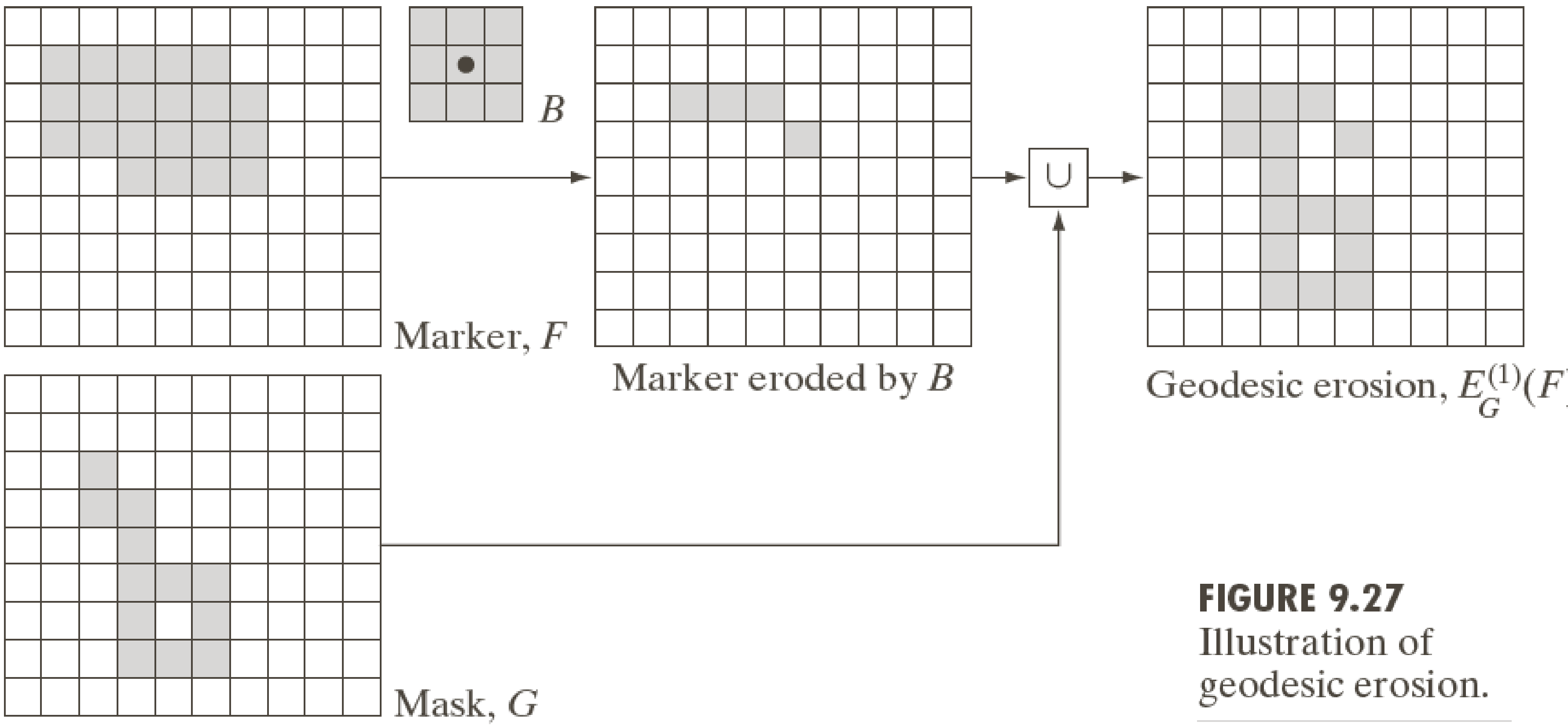


FIGURE 9.27
Illustration of geodesic erosion.

Morphological Reconstruction by Dilation

Morphological reconstruction by dialtion of a mask image G from a marker image F, denoted $R_G^D(F)$, is defined as the geodesic dilation of F with respect to G, iterated until stability is achieved; that is,

$$R_G^D(F) = D_G^{(k)}(F)$$

with k such that $D_G^{(k)}(F) = D_G^{(k-1)}(F)$.

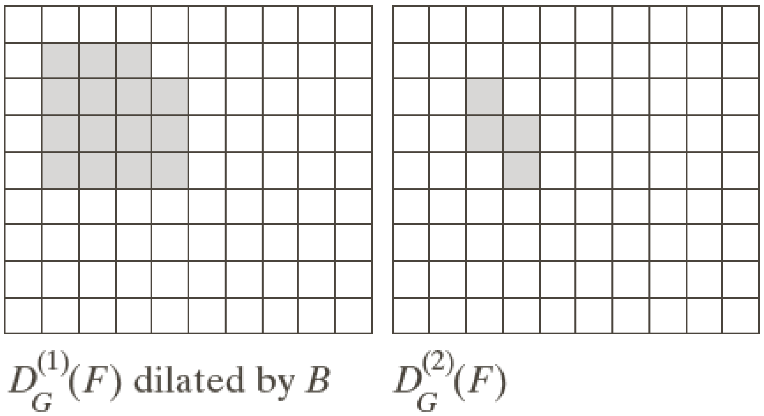


FIGURE 9.28 Illustration of morphological reconstruction by dilation. F, G, B and $D_G^{(1)}(F)$ are from Fig. 9.26.

Morphological Reconstruction by Erosion

Morphological reconstruction by erosion of a mask image G from a marker image F, denoted $R_G^E(F)$, is defined as the geodesic erosion of F with respect to G, iterated until stability is achieved; that is,

$$R_G^E(F) = E_G^{(k)}(F)$$

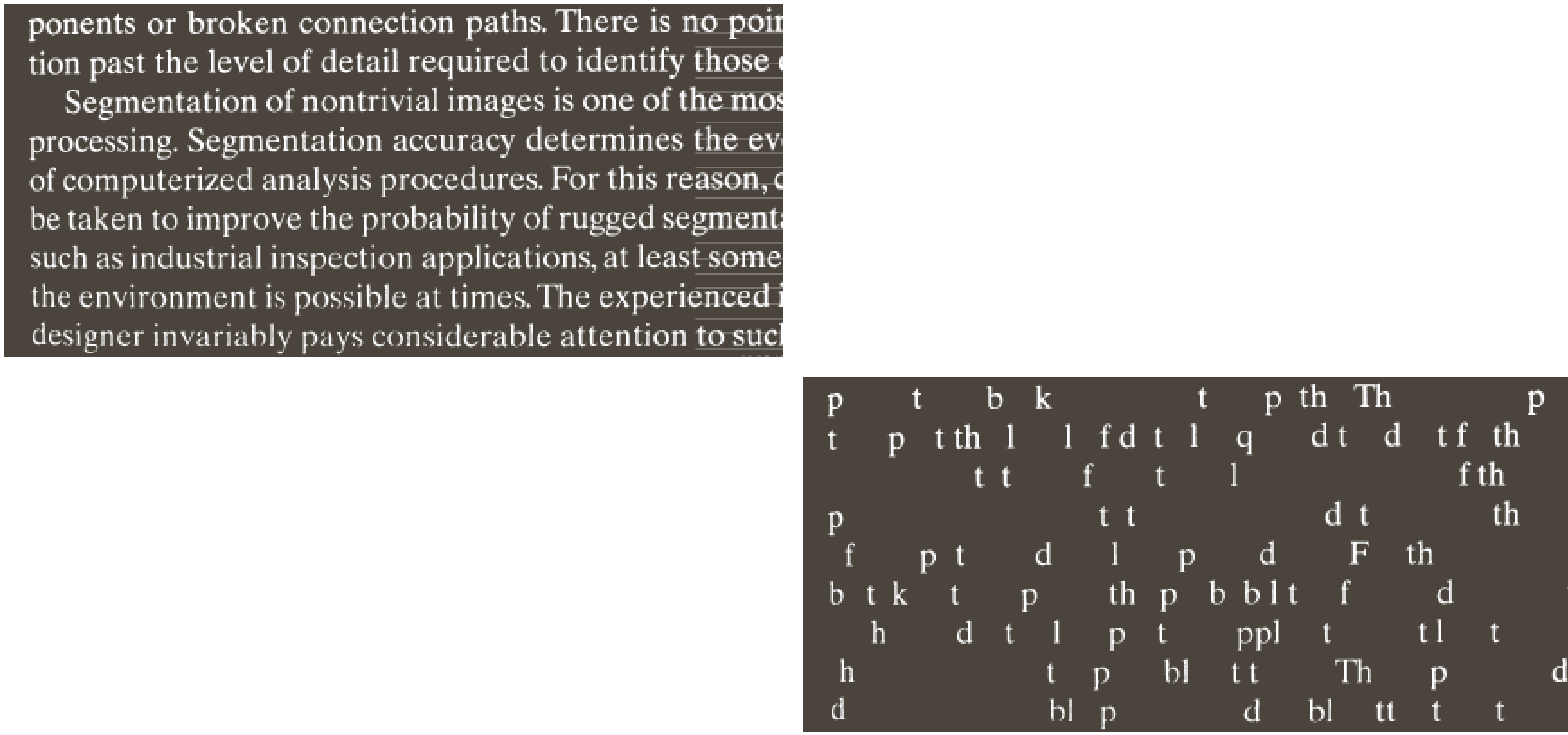
with k such that $E_G^{(k)}(F) = E_G^{(k-1)}(F)$.

Opening by Reconstruction

The opening by reconstruction of size n of an image F is defined as the reconstruction by dilation of F from the erosion of size n of F; that is

$$O_R^{(n)}(F) = R_F^D\left[\left(F \ominus nB\right)\right]$$

where $\left(F \ominus nB\right)$ indicates n erosions of F by B.


ponents or broken connection paths. There is no poi
tion past the level of detail required to identify those
Segmentation of nontrivial images is one of the mos
processing. Segmentation accuracy determines the ev
of computerized analysis procedures. For this reason,
be taken to improve the probability of rugged segment
such as industrial inspection applications, at least some
the environment is possible at times. The experienced
designer invariably pays considerable attention to suc

a b
c d

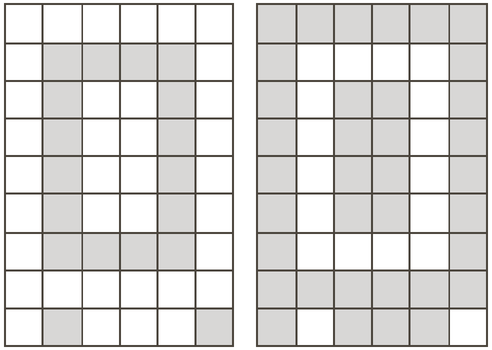
Filling Holes

Let $I(x, y)$ denote a binary image and suppose that we form a marker image F that is 0 everywhere, except at the image border, where it is set to $1 - I$; that is

$$F(x,y) = \begin{cases} 1 - I(x,y) & \text{if } (x,y) \text{ is on the border of } I \\ 0 & \text{otherwise} \end{cases}$$

then

$$H = \left[R_{I^c}^{D}(F) \right]^c$$

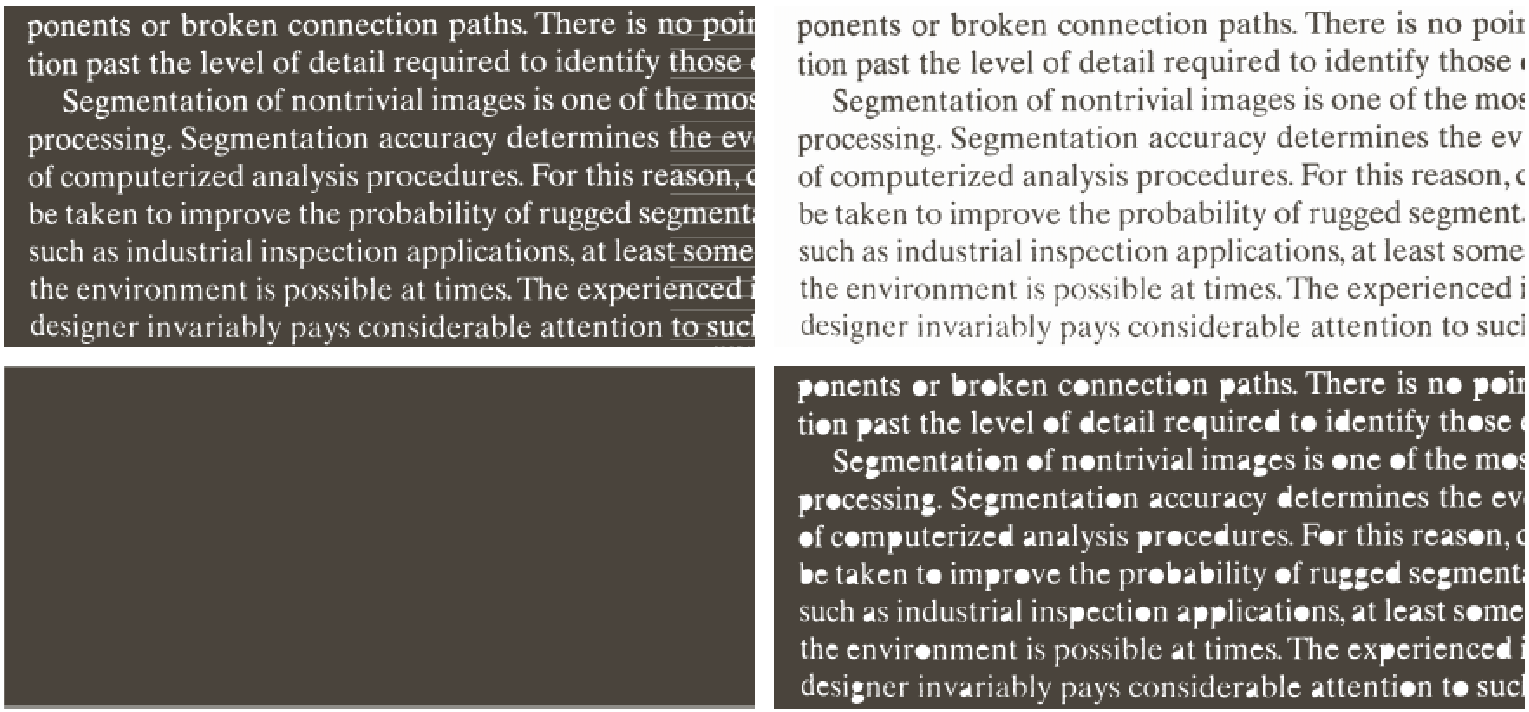


I $\quad I^c \quad F \quad F \oplus B \quad F \oplus B \cap I^c \quad H \quad H \cap I^c$

$\text{SE}: 3 \times 3 \ 1\text{s}.$

a b c d e f g

FIGURE 9.30 Illustration of hole filling on a simple image.



a b
c d

FIGURE 9.31 (a) Text image of size 918 × 2018 pixels. (b) Complement of (a) for use as a mask image. (c) Marker image. (d) Result of hole-filling using Eq. (9.5-29).

$$H = \left[R_{I^c}^{D}(F) \right]^c$$

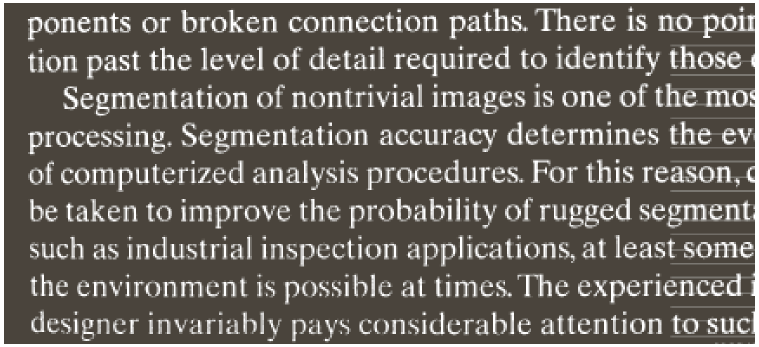
Border Clearing

It can be used to screen images so that only complete objects remain for further processing; it can be used as a singal that partial objects are present in the field of view.

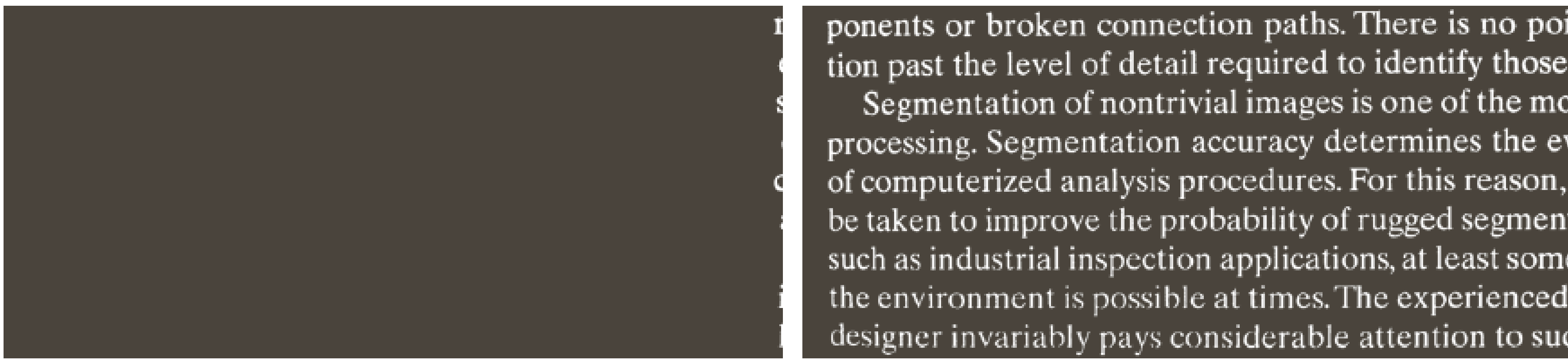
The original image is used as the mask and the following marker image:

$$F(x, y) = \begin{cases} I(x, y) & \text{if } (x, y) \text{ is on the border of } I \\ 0 & \text{otherwise} \end{cases}$$

$$X = I - R_I^D(F)$$



ponents or broken connection paths. There is no poi
tion past the level of detail required to identify those
Segmentation of nontrivial images is one of the mos
processing. Segmentation accuracy determines the ev
of computerized analysis procedures. For this reason,
be taken to improve the probability of rugged segment
such as industrial inspection applications, at least some
the environment is possible at times. The experienced
designer invariably pays considerable attention to suc



ponents or broken connection paths. There is no poi
tion past the level of detail required to identify those
Segmentation of nontrivial images is one of the mo
processing. Segmentation accuracy determines the ev
of computerized analysis procedures. For this reason,
be taken to improve the probability of rugged segment
such as industrial inspection applications, at least some
the environment is possible at times. The experienced
designer invariably pays considerable attention to suc

a b

FIGURE 9.32 Border clearing. (a) Marker image. (b) Image with no objects touching the border. The original image is Fig. 9.29(a).

Summary (1)

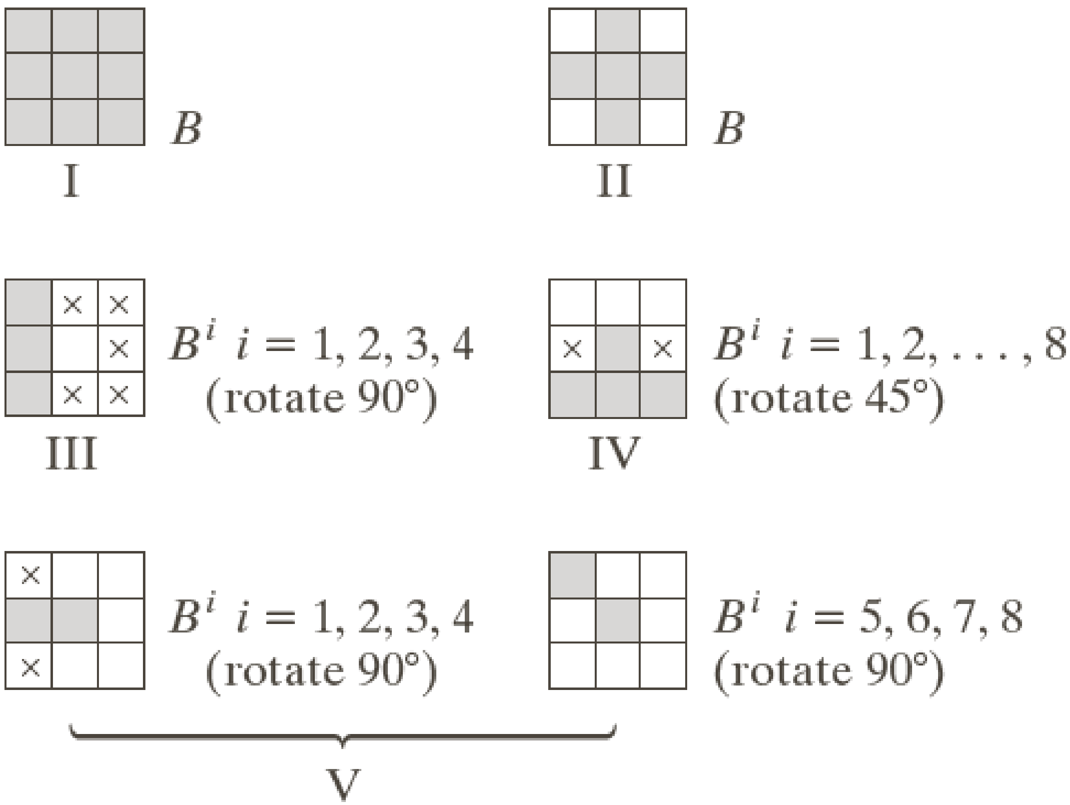


FIGURE 9.33 Five basic types of structuring elements used for binary morphology. The origin of each element is at its center and the ×'s indicate "don't care" values.

Summary (2)

Operation	Equation	Comments (The Roman numerals refer to the structuring elements in Fig. 9.33.)
Translation	$(B)_z = \{w \mid w = b + z, \text{ for } b \in B\}$	Translates the origin of B to point z.
Reflection	$\hat{B} = \{w \mid w = -b, \quad \text{for } b \in B\}$	Reflects all elements of B about the origin of this set.
Complement	$A^c = \{w \mid w \notin A\}$	Set of points not in A.
Difference	$A - B = \{w \mid w \in A, w \notin B\} = A \cap B^c$	Set of points that belong to A but not to B.
Dilation	$A \oplus B = \left\{z \mid (\hat{B}_z) \cap A \neq \varnothing\right\}$	"Expands" the boundary of A. (I)
Erosion	$A \ominus B = \left\{z \mid (B)_z \subseteq A\right\}$	"Contracts" the boundary of A. (I)
Opening	$A \circ B = (A \ominus B) \oplus B$	Smoothes contours, breaks narrow isthmuses, and eliminates small islands and sharp peaks. (I)

(*Continued*)

TABLE 9.1 Summary of morphological operations and their properties.

Operation	Equation	Comments (The Roman numerals refer to the structuring elements in Fig. 9.33.)
Closing	$A \bullet B = (A \oplus B) \ominus B$	Smoothes contours, fuses narrow breaks and long thin gulfs, and eliminates small holes. (I)
Hit-or-miss transform	$A \circledast B = (A \ominus B_1) \cap (A^c \ominus B_2) = (A \ominus B_1) - (A \oplus \hat{B}_2)$	The set of points (coordinates) at which, simultaneously, B_1 found a match ("hit") in A and B_2 found a match in A^c
Boundary extraction	$\beta(A) = A - (A \ominus B)$	Set of points on the boundary of set A. (I)
Hole filling	$X_k = (X_{k-1} \oplus B) \cap A^c$; $k = 1, 2, 3, \ldots$	Fills holes in A; X_0 = array of 0s with a 1 in each hole. (II)
Connected components	$X_k = (X_{k-1} \oplus B) \cap A$; $k = 1, 2, 3, \ldots$	Finds connected components in A; X_0 = array of 0s with a 1 in each connected component. (I)
Convex hull	$X_k^i = (X_{k-1}^i \circledast B^i) \cup A$; $i = 1, 2, 3, 4$; $k = 1, 2, 3, \ldots$; $X_0^i = A$; and $D^i = X_{\text{conv}}^i$	Finds the convex hull $C(A)$ of set A, where "conv" indicates convergence in the sense that $X_k^i = X_{k-1}^i$. (III)
Thinning	$A \otimes B = A - (A \circledast B) = A \cap (A \circledast B)^c$ $A \otimes \{B\} = ((\ldots((A \otimes B^1) \otimes B^2) \ldots) \otimes B^n)$ $\{B\} = \{B^1, B^2, B^3, \ldots, B^n\}$	Thins set A. The first two equations give the basic definition of thinning. The last equations denote thinning by a sequence of structuring elements. This method is normally used in practice. (IV)
Thickening	$A \odot B = A \cup (A \circledast B)$ $A \odot \{B\} = ((\ldots(A \odot B^1) \odot B^2 \ldots) \odot B^n)$	Thickens set A. (See preceding comments on sequences of structuring elements.) Uses IV with 0s and 1s reversed.
Skeletons	$S(A) = \bigcup_{k=0}^{K} S_k(A)$ $S_k(A) = \bigcup_{k=0}^{K} \{(A \ominus kB) - [(A \ominus kB) \circ B]\}$ Reconstruction of A: $A = \bigcup_{k=0}^{K} (S_k(A) \oplus kB)$	Finds the skeleton $S(A)$ of set A. The last equation indicates that A can be reconstructed from its skeleton subsets $S_k(A)$. In all three equations, K is the value of the iterative step after which the set A erodes to the empty set. The notation $(A \ominus kB)$ denotes the kth iteration of successive erosions of A by B. (I)

(Continued)

TABLE 9.1
(Continued)

TABLE 9.1 *(Continued)*

Operation	Equation	Comments (The Roman numerals refer to the structuring elements in Fig. 9.33.)
Pruning	$X_1 = A \otimes \{B\}$ $X_2 = \bigcup_{k=1}^{8}(X_1 \circledast B^k)$ $X_3 = (X_2 \oplus H) \cap A$ $X_4 = X_1 \cup X_3$	X_4 is the result of pruning set A. The number of times that the first equation is applied to obtain X_1 must be specified. Structuring elements V are used for the first two equations. In the third equation H denotes structuring element I.
Geodesic dilation of size 1	$D_G^{(1)}(F) = (F \oplus B) \cap G$	F and G are called the *marker* and *mask* images, respectively.
Geodesic dilation of size n	$D_G^{(n)}(F) = D_G^{(1)}\left[D_G^{(n-1)}(F)\right]$; $D_G^{(0)}(F) = F$	
Geodesic erosion of size 1	$E_G^{(1)}(F) = (F \ominus B) \cup G$	
Geodesic erosion of size n	$E_G^{(n)}(F) = E_G^{(1)}\left[E_G^{(n-1)}(F)\right]$; $E_G^{(0)}(F) = F$	
Morphological reconstruction by dilation	$R_G^D(F) = D_G^{(k)}(F)$	k is such that $D_G^{(k)}(F) = D_G^{(k+1)}(F)$
Morphological reconstruction by erosion	$R_G^E(F) = E_G^{(k)}(F)$	k is such that $E_G^{(k)}(F) = E_G^{(k+1)}(F)$
Opening by reconstruction	$O_R^{(n)}(F) = R_F^D[(F \ominus nB)]$	$(F \ominus nB)$ indicates n erosions of F by B.
Closing by reconstruction	$C_R^{(n)}(F) = R_F^E[(F \oplus nB)]$	$(F \oplus nB)$ indicates n dilations of F by B.
Hole filling	$H = \left[R_{I^c}^D(F)\right]^c$	H is equal to the input image I, but with all holes filled. See Eq. (9.5-28) for the definition of the marker image F.
Border clearing	$X = I - R_I^D(F)$	X is equal to the input image I, but with all objects that touch (are connected to) the boundary removed. See Eq. (9.5-30) for the definition of the marker image F.

Gray-Scale Morphology

$$f(x, y): \text{ gray-scale image}$$

$$b(x, y): \text{ structuring element}$$

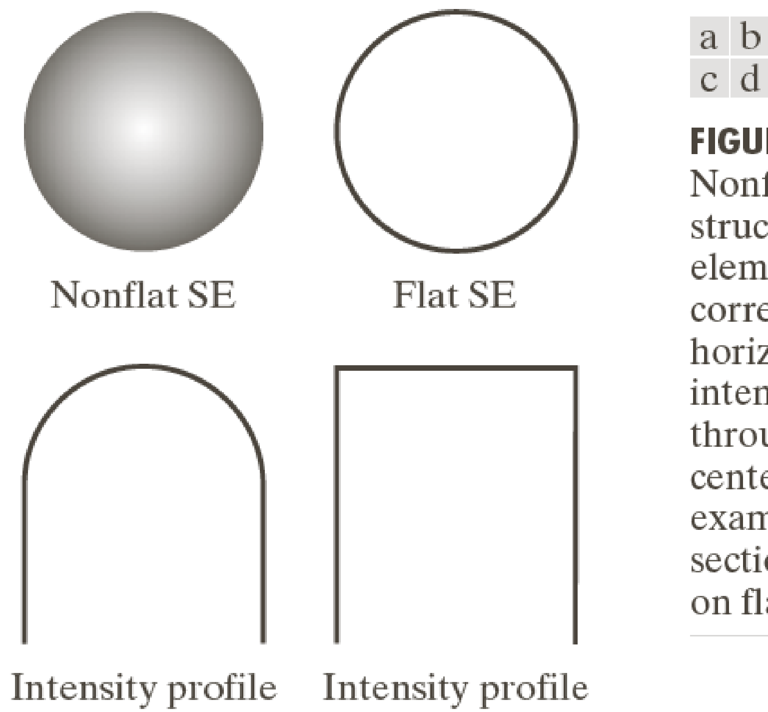
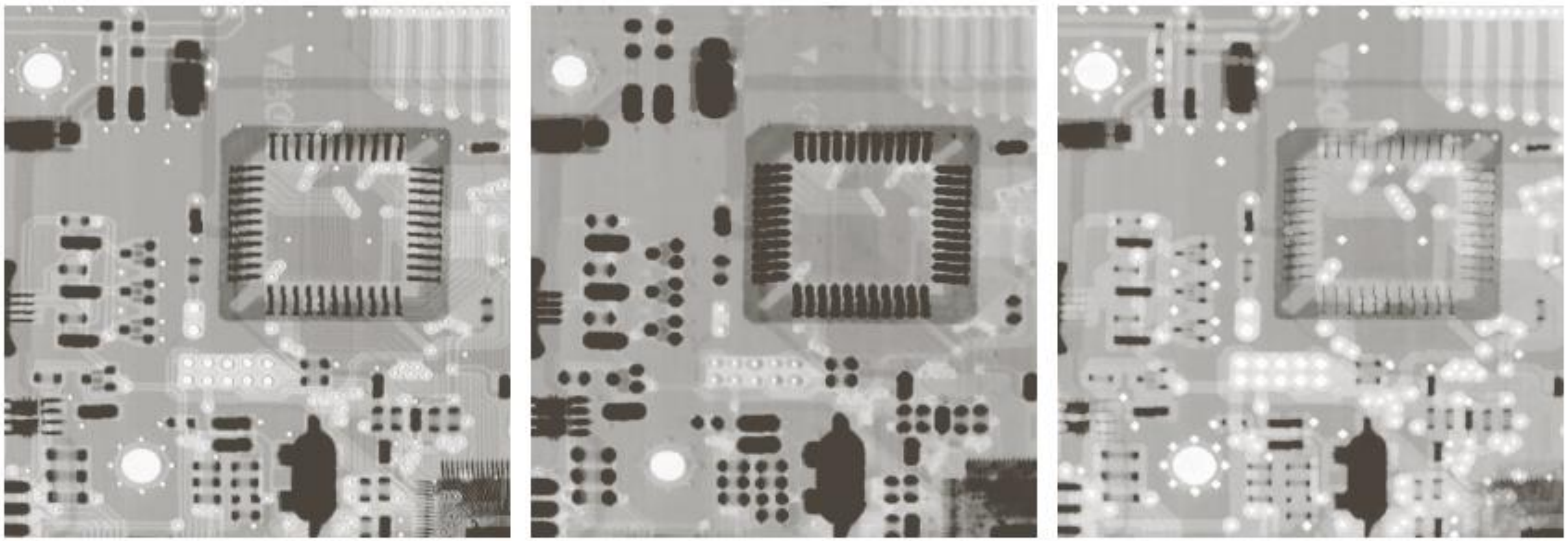


FIGURE 9.34 Nonflat and flat structuring elements, and corresponding horizontal intensity profiles through their center. All examples in this section are based on flat SEs.

Gray-Scale Morphology: Erosion and Dilation by Flat Structuring

$$\left[f \ominus b \right](x, y) = \min_{(s,t)\in b} \left\{ f(x+s, y+t) \right\}$$

$$\left[f \oplus b \right](x, y) = \max_{(s,t)\in b} \left\{ f(x-s, y-t) \right\}$$



a b c

FIGURE 9.35 (a) A gray-scale X-ray image of size 448 × 425 pixels. (b) Erosion using a flat disk SE with a radius of two pixels. (c) Dilation using the same SE. (Original image courtesy of Lixi, Inc.)

Gray-Scale Morphology: Erosion and Dilation by Nonflat Structuring

$$\left[f \ominus b_N \right](x, y) = \min_{(s,t)\in b} \left\{ f(x+s, y+t) - b_N(s,t) \right\}$$

$$\left[f \oplus b_N \right](x, y) = \max_{(s,t)\in b} \left\{ f(x-s, y-t) + b_N(s,t) \right\}$$

Duality: Erosion and Dilation

$$\left[f \ominus b \right]^c (x, y) = \left(f^c \oplus \hat{b} \right)(x, y)$$

$$\text{where } f^c = -f(x, y) \text{ and } \hat{b} = b(-x, -y)$$

$$\left[f \ominus b \right]^c = \left(f^c \oplus \hat{b} \right)$$

$$\left(f \oplus b \right)^c = \left(f^c \ominus \hat{b} \right)$$

Opening and Closing

$$f \circ b = \left(f \ominus b \right) \oplus b$$

$$f \bullet b = \left(f \oplus b \right) \ominus b$$

$$\left(f \square b \right)^{c} = f^{c} \circ \hat{b} = -f \circ \hat{b}$$

$$\left(f \circ b \right)^{c} = f^{c} \square \hat{b} = -f \square \hat{b}$$

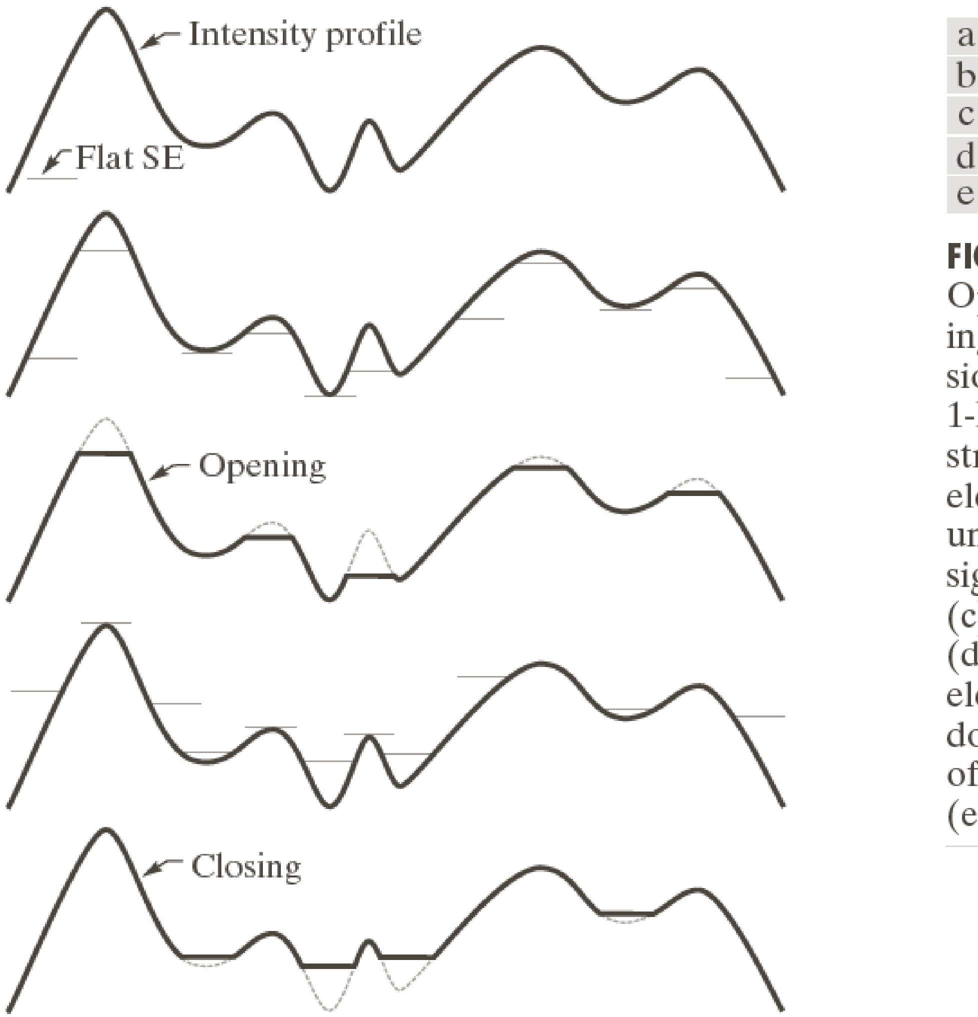


FIGURE 9.36 Opening and closing in one dimension. (a) Original 1-D signal. (b) Flat structuring element pushed up underneath the signal. (c) Opening. (d) Flat structuring element pushed down along the top of the signal. (e) Closing.

Properties of Gray-scale Opening

$$(a)\quad f \circ b \dashv f$$

$$(b)\quad \text{if } f_1 \dashv f_2,\ \text{then } \left(f_1 \circ b\right) \dashv \left(f_2 \circ b\right)$$

$$(c)\quad \left(f \circ b\right) \circ b = f \circ b$$

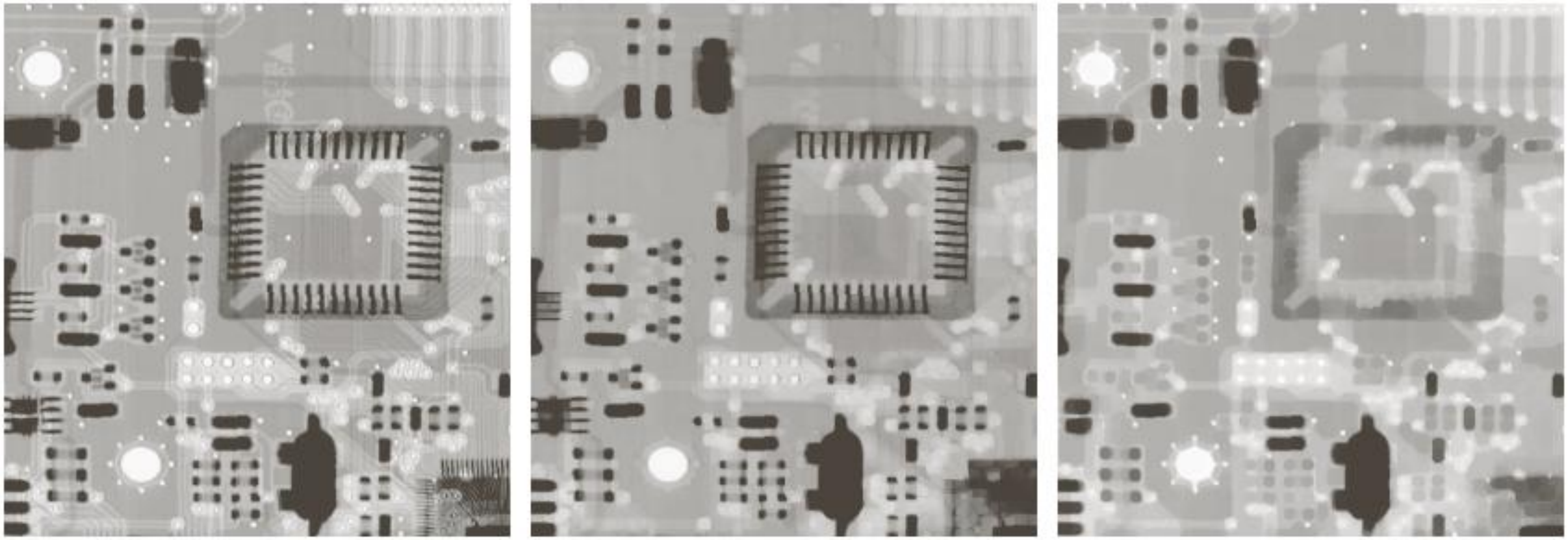
where $e \dashv r$ denotes e is a subset of r and also $e(x, y) \le r(x, y)$.

Properties of Gray-scale Closing

$$(a)\quad f \dashv f \bullet b$$

$$(b)\quad \text{if } f_1 \dashv f_2,\ \text{then } \left(f_1 \bullet b \right) \dashv \left(f_2 \bullet b \right)$$

$$(c)\quad \left(f \bullet b \right) \bullet b = f \bullet b$$



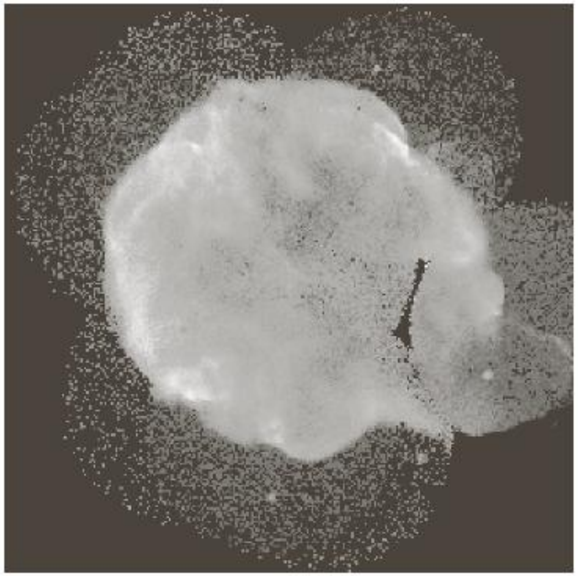
a b c

Morphological Smoothing

- Opening suppresses bright details smaller than the specified SE, and closing suppresses dark details.

- Opening and closing are used often in combination as morphological filters for image smoothing and noise removal.

Morphological Smoothing



a	b
c	d

FIGURE 9.38 (a) 566 × 566 image of the Cygnus Loop supernova, taken in the X-ray band by NASA's Hubble Telescope. (b)–(d) Results of performing opening and closing sequences on the original image with disk structuring elements of radii, 1, 3, and 5, respectively. (Original image courtesy of NASA.)

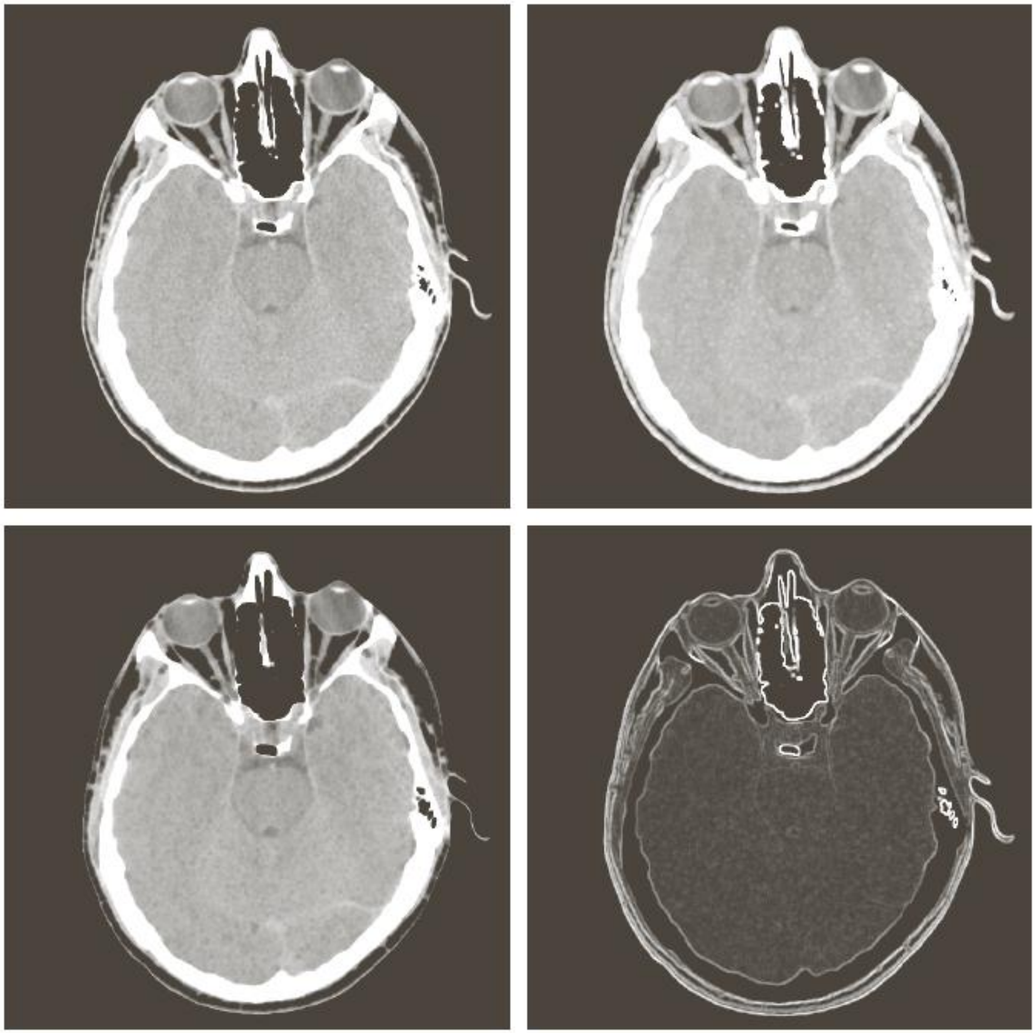
Morphological Gradient

- Dilation and erosion can be used in combination with image subtraction to obtain the morphological gradient of an image, denoted by g,

$$g = (f \oplus b) - (f \ominus b)$$

- The edges are enhanced and the contribution of the homogeneous areas are suppressed, thus producing a “derivative-like” (gradient) effect.

Morphological Gradient



a	b
c	d

FIGURE 9.39 (a) 512 × 512 image of a head CT scan. (b) Dilation. (c) Erosion. (d) Morphological gradient, computed as the difference between (b) and (c). (Original image courtesy of Dr. David R. Pickens, Vanderbilt University.)

Top-hat and Bottom-hat Transformations

- The top-hat transformation of a grayscale image f is defined as f minus its opening:

$$T_{hat}(f) = f - (f \circ b)$$

- The bottom-hat transformation of a grayscale image f is defined as its closing minus f:

$$B_{hat}(f) = (f \bullet b) - f$$

Top-hat and Bottom-hat Transformations

- One of the principal applications of these transformations is in removing objects from an image by using structuring element in the opening or closing operation

Example of Using Top-hat Transformation in Segmentation

a	b
c	d	e

FIGURE 9.40 Using the top-hat transformation for *shading correction*. (a) Original image of size 600 × 600 pixels. (b) Thresholded image. (c) Image opened using a disk SE of radius 40. (d) Top-hat transformation (the image minus its opening). (e) Thresholded top-hat image.

Granulometry

- Granulometry deals with determining the size of distribution of particles in an image
- Opening operations of a particular size should have the most effect on regions of the input image that contain particles of similar size
- For each opening, the sum (**surface area**) of the pixel values in the opening is computed

Example

a	b	c
d	e	f

FIGURE 9.41 (a) 531 × 675 image of wood dowels. (b) Smoothed image. (c)–(f) Openings of (b) with disks of radii equal to 10, 20, 25, and 30 pixels, respectively. (Original image courtesy of Dr. Steve Eddins, The MathWorks, Inc.)

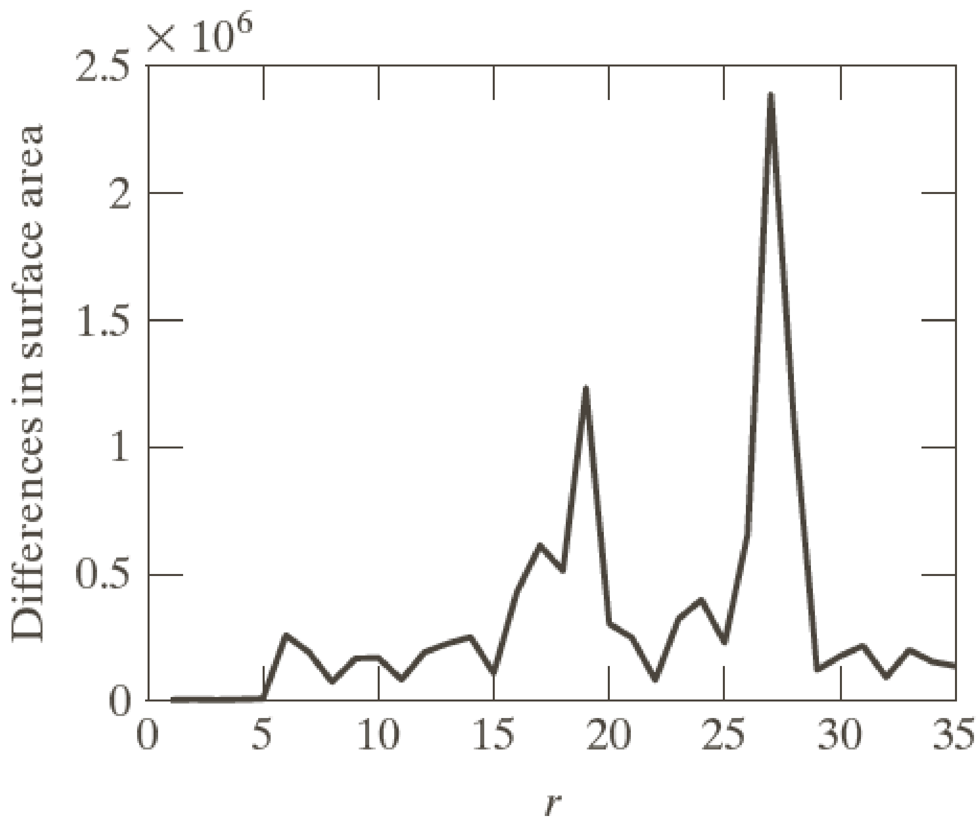
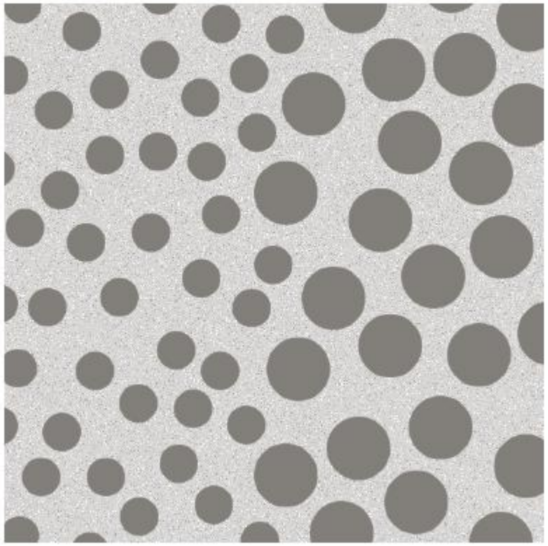


FIGURE 9.42 Differences in surface area as a function of SE disk radius, *r*. The two peaks are indicative of two dominant particle sizes in the image.

Textual Segmentation

- Segmentation: the process of subdividing an image into regions.

Textual Segmentation



a b
c d

FIGURE 9.43 Textural segmentation. (a) A 600 × 600 image consisting of two types of blobs. (b) Image with small blobs removed by closing (a). (c) Image with light patches between large blobs removed by opening (b). (d) Original image with boundary between the two regions in (c) superimposed. The boundary was obtained using a morphological gradient operation.

Gray-Scale Morphological Reconstruction (1)

- Let f and g denote the marker and mask image with the same size, respectively and f $\leq$ g.

  The geodesic dilation of size 1 of f with respect to g is defined as

$$D_g^{(1)}(f) = \left(f \oplus g\right) \wedge g$$

where $\wedge$ denotes the point-wise minimum operator.

The geodesic dilation of size n of f with respect to g is defined as

$$D_g^{(n)}(f) = D_g^{(1)}\left[D_g^{(n-1)}(f)\right] \text{ with } D_g^{(0)}(f) = f$$

Gray-Scale Morphological Reconstruction (2)

- The geodesic erosion of size 1 of f with respect to g is defined as

$$E_g^{(1)}(f) = \left(f \ominus g\right) \vee g$$

where $\vee$ denotes the point-wise maximum operator.

The geodesic erosion of size n of f with respect to g is defined as

$$E_g^{(n)}(f) = E_g^{(1)}\left[E_g^{(n-1)}(f)\right] \text{ with } E_g^{(0)}(f) = f$$

Gray-Scale Morphological Reconstruction (3)

- The morphological reconstruction by dilation of a gray-scale mask image g by a gray-scale marker image f, is defined as the geodesic dilation of f with respect to g, iterated until stability is reached, that is,

$$R_g^D(f) = D_g^{(k)}(f)$$

$$\text{with k such that } D_g^{(k)}(f) = D_g^{(k+1)}(f)$$

The morphological reconstruction by erosion of g by f is defined as

$$R_g^E(f) = E_g^{(k)}(f)$$

$$\text{with k such that } E_g^{(k)}(f) = E_g^{(k+1)}(f)$$

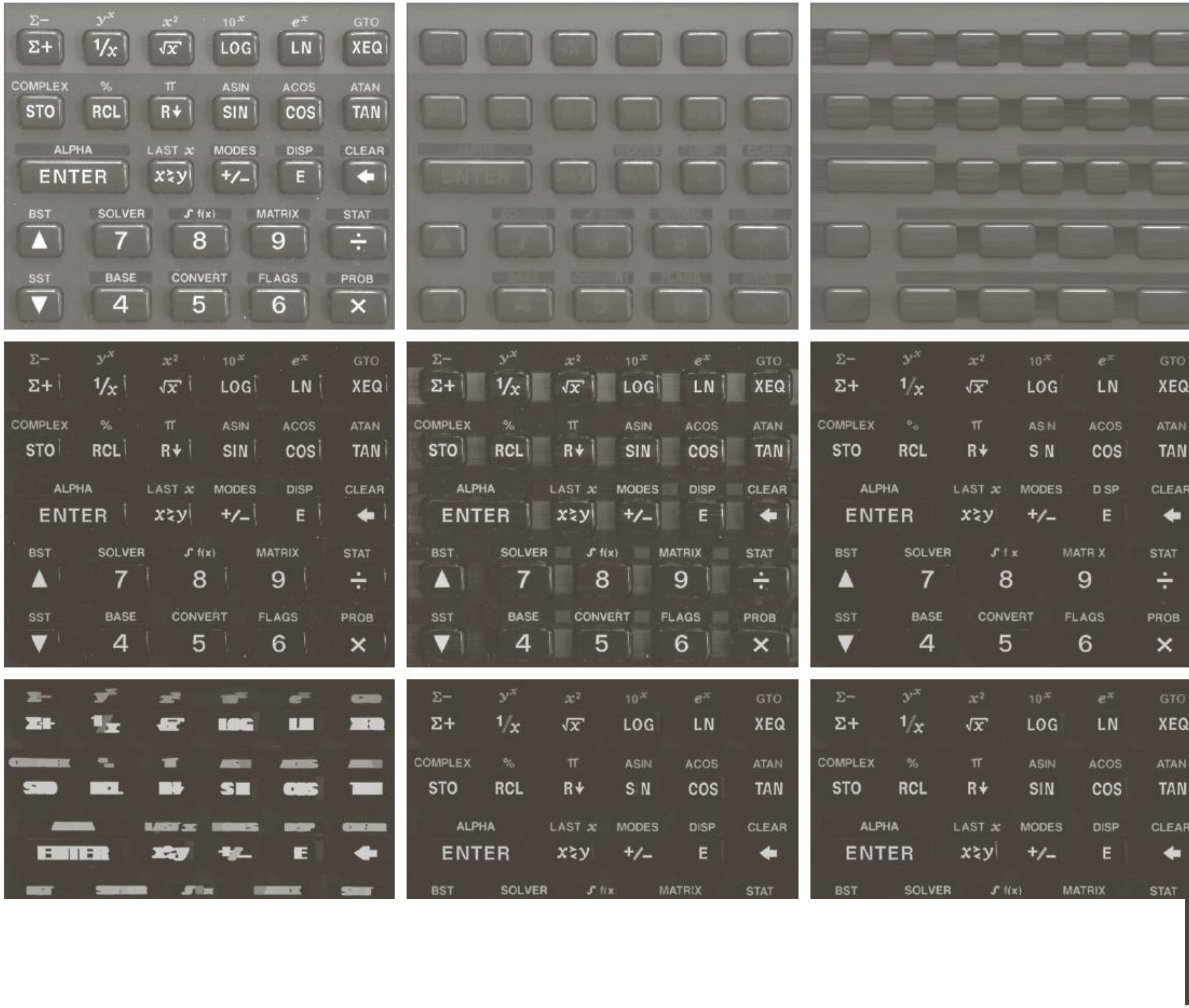
Gray-Scale Morphological Reconstruction (4)

- The opening by reconstruction of size n of an image f is defined as the reconstruction by dilation of f from the erosion of size n of f; that is,

$$O_R^{(n)}(f) = R_f^D\left[f \ominus nb\right]$$

The closing by reconstruction of size n of an image f is defined as the reconstruction by erosion of f from the dilation of size n of f; that is,

$$C_R^{(n)}(f) = R_f^E\left[f \oplus nb\right]$$



a b c
d e f
g h i

FIGURE 9.44 (a) Original image of size 1134 × 1360 pixels. (b) Opening by reconstruction of (a) using a horizontal line 71 pixels long in the erosion. (c) Opening of (a) using the same line. (d) Top-hat by reconstruction. (e) Top-hat. (f) Opening by reconstruction of (d) using a horizontal line 11 pixels long. (g) Dilation of (f) using a horizontal line 21 pixels long. (h) Minimum of (d) and (g). (i) Final reconstruction result. (Images courtesy of Dr. Steve Eddins, The MathWorks, Inc.)

Steps in the Example

1. Opening by reconstruction of the original image using a horizontal line of size 1x71 pixels in the erosion operation

$$O_R^{(n)}(f) = R_f^D\left[f \ominus nb\right]$$

2. Subtract the opening by reconstruction from original image

$$f' = f - O_R^{(n)}(f)$$

3. Opening by reconstruction of the f' using a vertical line of size 11x1 pixels

$$f1 = O_R^{(n)}(f') = R_f^D\left[f' \ominus nb'\right]$$

4. Dilate f1 with a line SE of size 1x21, get f2.

Steps in the Example

5. Calculate the minimum between the dilated image f2 and and f', get f3.

6. By using f3 as a marker and the dilated image f2 as the mask,

$$R_{f2}^{D}(f3) = D_{f2}^{(k)}(f3)$$

$$\text{with k such that } D_{f2}^{(k)}(f3) = D_{f2}^{(k+1)}(f3)$$