SNS COLLEGE OF TECHNOLOGY
(An Autonomous Institution)
Coimbatore-35

23ECT202 – SIGNALS & SYSTEMS

DISCRETE FOURIER SERIES

Discrete Fourier Series

- Given a periodic sequence $\tilde{x}[n]$ with period N so that

$$\tilde{x}[n] = \tilde{x}[n + rN]$$

- The Fourier series representation can be written as

$$\tilde{x}[n] = \frac{1}{N}\sum_{k}\tilde{X}[k]e^{j(2\pi/N)kn}$$

- The Fourier series representation of continuous-time periodic signals require infinite many complex exponentials
- Not that for discrete-time periodic signals we have

$$e^{j(2\pi/N)(k+mN)n} = e^{j(2\pi/N)kn}e^{j(2\pi mn)} = e^{j(2\pi/N)kn}$$

- Due to the periodicity of the complex exponential we only need N exponentials for discrete time Fourier series

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]e^{j(2\pi/N)kn}$$

Discrete Fourier Series Pair

- A periodic sequence in terms of Fourier series coefficients

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]e^{j(2\pi/N)kn}$$

- The Fourier series coefficients can be obtained via

$$\tilde{X}[k] = \sum_{n=0}^{N-1}\tilde{x}[n]e^{-j(2\pi/N)kn}$$

- For convenience we sometimes use

$$W_N = e^{-j(2\pi/N)}$$

- Analysis equation

$$\tilde{X}[k] = \sum_{n=0}^{N-1}\tilde{x}[n]W_N^{kn}$$

- Synthesis equation

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]W_N^{-kn}$$

Example 1

- DFS of a periodic impulse train

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} \delta[n - rN] = \begin{cases} 1 & n = rN \\ 0 & \text{else} \end{cases}$$

- Since the period of the signal is N

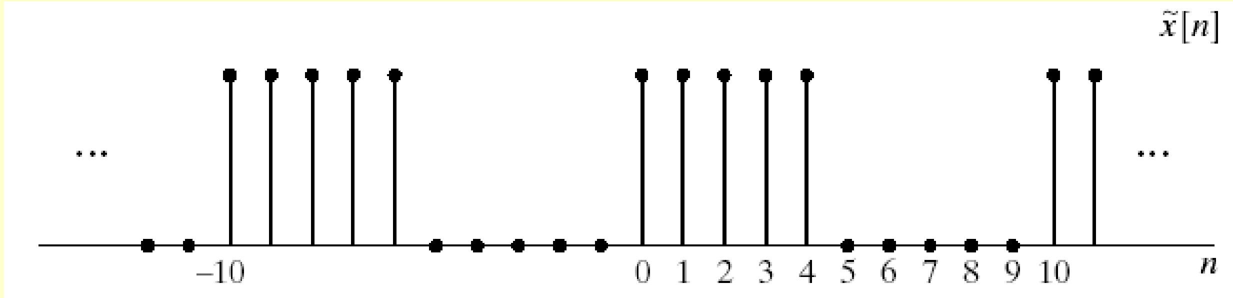
$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j(2\pi/N)kn} = \sum_{n=0}^{N-1} \delta[n] e^{-j(2\pi/N)kn} = e^{-j(2\pi/N)k0} = 1$$

- We can represent the signal with the DFS coefficients as

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} \delta[n - rN] = \frac{1}{N} \sum_{k=0}^{N-1} e^{j(2\pi/N)kn}$$

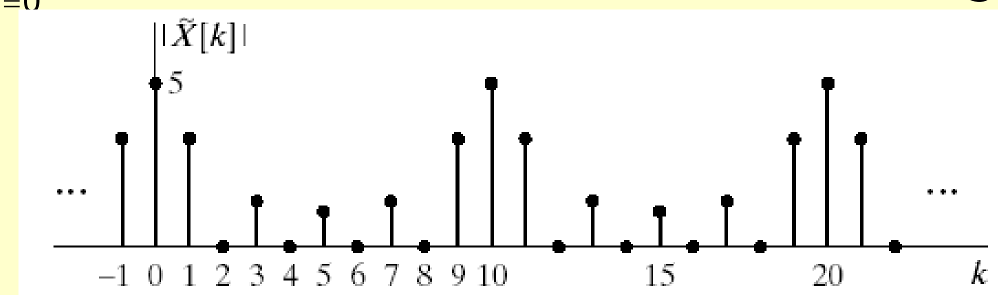
Example 2

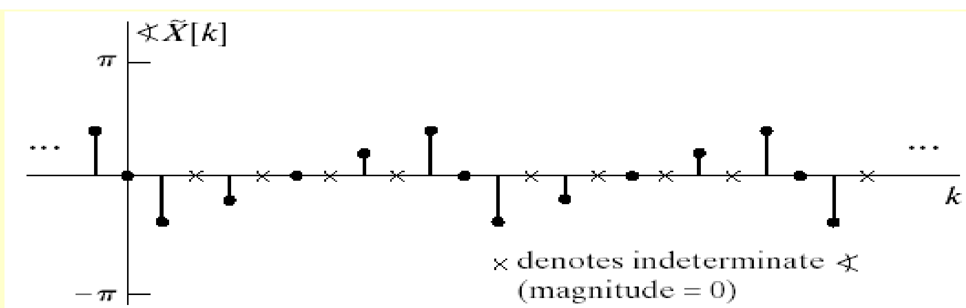
- DFS of an periodic rectangular pulse train



- The DFS coefficients

$$\tilde{X}[k] = \sum_{n=0}^{4} e^{-j(2\pi/10)kn} = \frac{1-e^{-j(2\pi/10)k5}}{1-e^{-j(2\pi/10)k}} = e^{-j(4\pi k/10)}\frac{\sin(\pi k/2)}{\sin(\pi k/10)}$$





Properties of DFS

- Linearity

$$\tilde{x}_1[n] \overset{DFS}{\longleftrightarrow} \tilde{X}_1[k]$$
$$\tilde{x}_2[n] \overset{DFS}{\longleftrightarrow} \tilde{X}_2[k]$$
$$a\tilde{x}_1[n] + b\tilde{x}_2[n] \overset{DFS}{\longleftrightarrow} a\tilde{X}_1[k] + b\tilde{X}_2[k]$$

- Shift of a Sequence

$$\tilde{x}[n] \overset{DFS}{\longleftrightarrow} \tilde{X}[k]$$
$$\tilde{x}[n-m] \overset{DFS}{\longleftrightarrow} e^{-j2\pi km/N}\tilde{X}[k]$$
$$e^{j2\pi nm/N}\tilde{x}[n] \overset{DFS}{\longleftrightarrow} \tilde{X}[k-m]$$

- Duality

$$\tilde{x}[n] \overset{DFS}{\longleftrightarrow} \tilde{X}[k]$$
$$\tilde{X}[n] \overset{DFS}{\longleftrightarrow} N\tilde{x}[-k]$$

Symmetry Properties

Periodic Sequence (Period N)	DFS Coefficients (Period N)
1. $\tilde{x}[n]$	$\tilde{X}[k]$ periodic with period N
2. $\tilde{x}_1[n], \tilde{x}_2[n]$	$\tilde{X}_1[k], \tilde{X}_2[k]$ periodic with period N
3. $a\tilde{x}_1[n] + b\tilde{x}_2[n]$	$a\tilde{X}_1[k] + b\tilde{X}_2[k]$
4. $\tilde{X}[n]$	$N\tilde{x}[-k]$
5. $\tilde{x}[n-m]$	$W_N^{km}\tilde{X}[k]$
6. $W_N^{-\ell n}\tilde{x}[n]$	$\tilde{X}[k-\ell]$
7. $\sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m]$ (periodic convolution)	$\tilde{X}_1[k]\tilde{X}_2[k]$
8. $\tilde{x}_1[n]\tilde{x}_2[n]$	$\frac{1}{N}\sum_{\ell=0}^{N-1} \tilde{X}_1[\ell]\tilde{X}_2[k-\ell]$ (periodic convolution)
9. $\tilde{x}^*[n]$	$\tilde{X}^*[-k]$

Symmetry Properties Cont'd

Periodic Sequence (Period N)	DFS Coefficients (Period N)
10. $\tilde{x}^*[-n]$	$\tilde{X}^*[k]$
11. $\mathcal{R}e\{\tilde{x}[n]\}$	$\tilde{X}_e[k] = \frac{1}{2}(\tilde{X}[k] + \tilde{X}^*[-k])$
12. $j\mathcal{J}m\{\tilde{x}[n]\}$	$\tilde{X}_o[k] = \frac{1}{2}(\tilde{X}[k] - \tilde{X}^*[-k])$
13. $\tilde{x}_e[n] = \frac{1}{2}(\tilde{x}[n] + \tilde{x}^*[-n])$	$\mathcal{R}e\{\tilde{X}[k]\}$
14. $\tilde{x}_o[n] = \frac{1}{2}(\tilde{x}[n] - \tilde{x}^*[-n])$	$j\mathcal{J}m\{\tilde{X}[k]\}$
Properties 15–17 apply only when $x[n]$ is real.	
15. Symmetry properties for $\tilde{x}[n]$ real.	$\begin{cases} \tilde{X}[k] = \tilde{X}^*[-k] \\ \mathcal{R}e\{\tilde{X}[k]\} = \mathcal{R}e\{\tilde{X}[-k]\} \\ \mathcal{J}m\{\tilde{X}[k]\} = -\mathcal{J}m\{\tilde{X}[-k]\} \\ \lvert\tilde{X}[k]\rvert = \lvert\tilde{X}[-k]\rvert \\ \sphericalangle\tilde{X}[k] = -\sphericalangle\tilde{X}[-k] \end{cases}$
16. $\tilde{x}_e[n] = \frac{1}{2}(\tilde{x}[n] + \tilde{x}[-n])$	$\mathcal{R}e\{\tilde{X}[k]\}$
17. $\tilde{x}_0[n] = \frac{1}{2}(\tilde{x}[n] - \tilde{x}[-n])$	$j\mathcal{J}m\{\tilde{X}[k]\}$

Periodic Convolution

- Take two periodic sequences

$$\tilde{x}_1[n] \overset{DFS}{\longleftrightarrow} \tilde{X}_1[k]$$
$$\tilde{x}_2[n] \overset{DFS}{\longleftrightarrow} \tilde{X}_2[k]$$

- Let's form the product

$$\tilde{X}_3[k] = \tilde{X}_1[k]\tilde{X}_2[k]$$

- The periodic sequence with given DFS can be written as

$$\tilde{x}_3[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m]$$

- Periodic convolution is commutative

$$\tilde{x}_3[n] = \sum_{m=0}^{N-1} \tilde{x}_2[m]\tilde{x}_1[n-m]$$

Periodic Convolution Cont'd

$$\tilde{x}_3[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m]$$

- Substitute periodic convolution into the DFS equation

$$\tilde{X}_3[k] = \sum_{n=0}^{N-1}\left(\sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m]\right)W_N^{kn}$$

- Interchange summations

$$\tilde{X}_3[k] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\left(\sum_{n=0}^{N-1} \tilde{x}_2[n-m]W_N^{kn}\right)$$

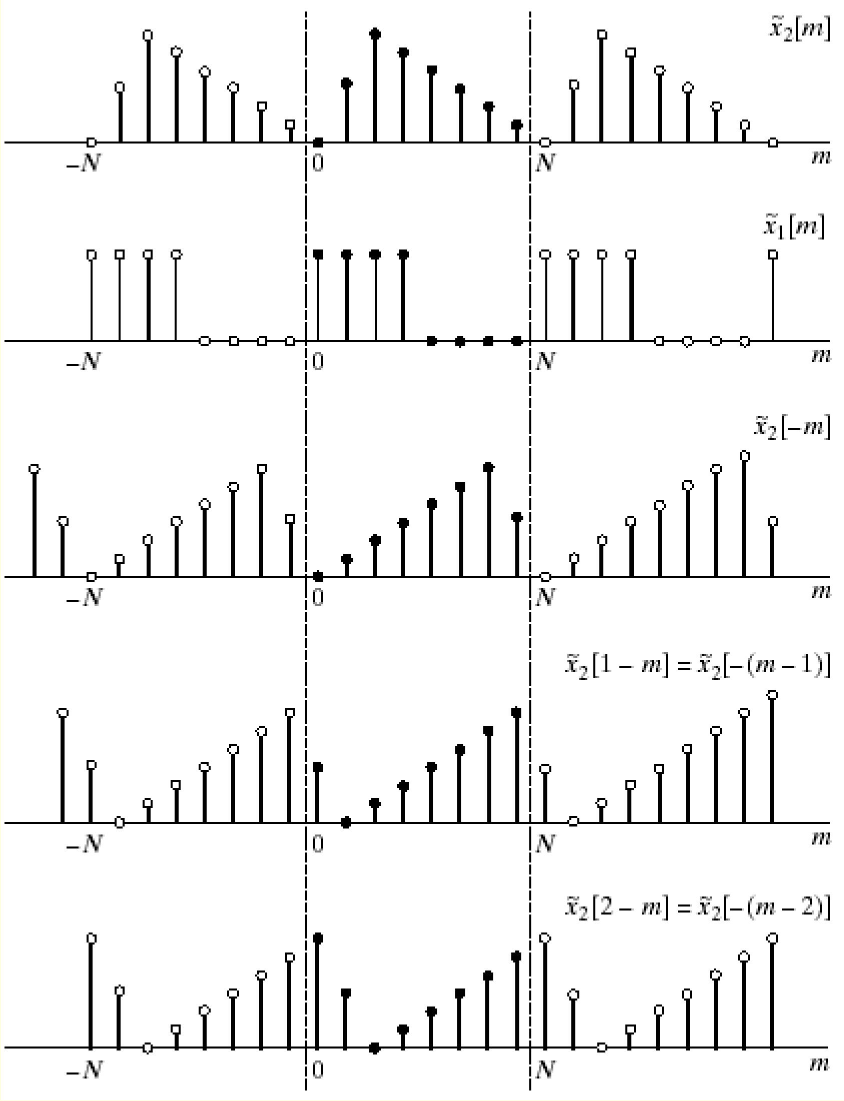
- The inner sum is the DFS of shifted sequence

$$\sum_{n=0}^{N-1} \tilde{x}_2[n-m]W_N^{kn} = W_N^{km}\tilde{X}_2[k]$$

- Substituting

$$\tilde{X}_3[k] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\left(\sum_{n=0}^{N-1} \tilde{x}_2[n-m]W_N^{kn}\right) = \sum_{m=0}^{N-1} \tilde{x}_1[m]W_N^{km}\tilde{X}_2[k] = \tilde{X}_1[k]\tilde{X}_2[k]$$

Graphical Periodic Convolution



The Fourier Transform of Periodic Signals

- Periodic sequences are not absolute or square summable
  - Hence they don't have a Fourier Transform
- We can represent them as sums of complex exponentials: DFS
- We can combine DFS and Fourier transform
- Fourier transform of periodic sequences
  - Periodic impulse train with values proportional to DFS coefficients

$$\tilde{X}\left(e^{j\omega}\right) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{N} \tilde{X}[k]\,\delta\left(\omega - \frac{2\pi k}{N}\right)$$

  - This is periodic with 2π since DFS is periodic
- The inverse transform can be written as

$$\frac{1}{2\pi}\int_{0-\varepsilon}^{2\pi-\varepsilon} \tilde{X}\left(e^{j\omega}\right) e^{j\omega n} d\omega = \frac{1}{2\pi}\int_{0-\varepsilon}^{2\pi-\varepsilon} \sum_{k=-\infty}^{\infty} \frac{2\pi}{N} \tilde{X}[k]\,\delta\left(\omega - \frac{2\pi k}{N}\right) e^{j\omega n} d\omega$$

$$\frac{1}{N}\sum_{k=-\infty}^{\infty} \tilde{X}[k] \int_{0-\varepsilon}^{2\pi-\varepsilon} \delta\left(\omega - \frac{2\pi k}{N}\right) e^{j\omega n} d\omega = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k]\, e^{j\frac{2\pi k}{N} n}$$

Example

- Consider the periodic impulse train

$$\tilde{p}[n] = \sum_{r=-\infty}^{\infty} \delta[n - rN]$$

- The DFS was calculated previously to be

$$\tilde{P}[k] = 1 \quad \text{for all } k$$

- Therefore the Fourier transform is

$$\tilde{P}(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{N} \delta\left(\omega - \frac{2\pi k}{N}\right)$$

Relation between Finite-length and Periodic Signals

- Consider finite length signal x[n] spanning from 0 to N-1
- Convolve with periodic impulse train

$$\tilde{x}[n] = x[n] * \tilde{p}[n] = x[n] * \sum_{r=-\infty}^{\infty} \delta[n - rN] = \sum_{r=-\infty}^{\infty} x[n - rN]$$

- The Fourier transform of the periodic sequence is

$$\tilde{X}(e^{j\omega}) = X(e^{j\omega})\tilde{P}(e^{j\omega}) = X(e^{j\omega})\sum_{k=-\infty}^{\infty} \frac{2\pi}{N} \delta\left(\omega - \frac{2\pi k}{N}\right)$$

$$\tilde{X}(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{N} X\left(e^{j\frac{2\pi k}{N}}\right)\delta\left(\omega - \frac{2\pi k}{N}\right)$$

- This implies that

$$\tilde{X}[k] = X\left(e^{j\frac{2\pi k}{N}}\right) = X(e^{j\omega})\Big|_{\omega=\frac{2\pi k}{N}}$$

- DFS coefficients of a periodic signal can be thought as equally spaced samples of the Fourier transform of one period

Example

- Consider the following sequence

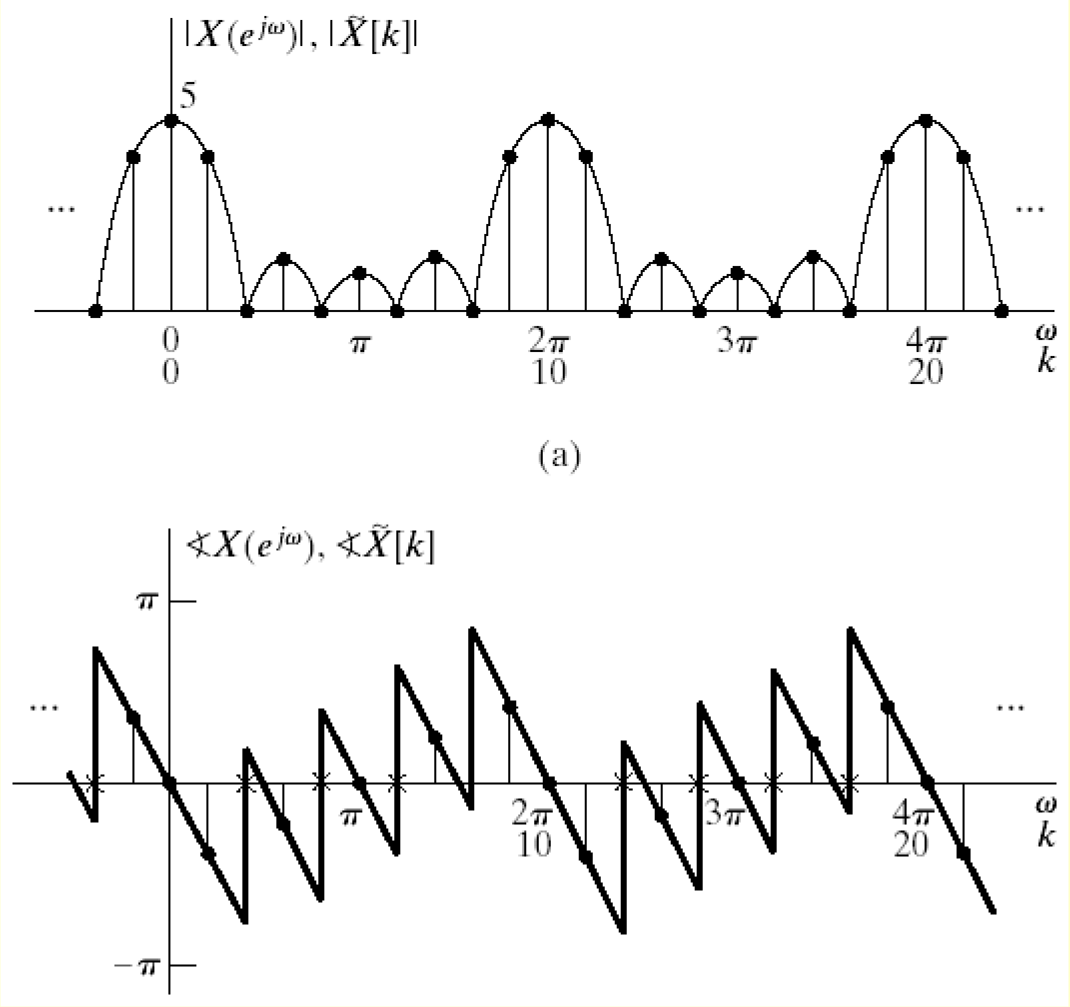
$$x[n] = \begin{cases} 1 & 0 \le n \le 4 \\ 0 & \text{else} \end{cases}$$

- The Fourier transform

$$X\left(e^{j\omega}\right) = e^{-j2\omega}\frac{\sin(5\omega/2)}{\sin(\omega/2)}$$

- The DFS coefficients

$$\tilde{X}[k] = e^{-j(4\pi k/10)}\frac{\sin(\pi k/2)}{\sin(\pi k/10)}$$



THANK YOU