1) Determine the Fourier transform of $x(n) = a^n u(n)$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} a^n u(n)\, e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} a^n e^{-j\omega n} \Rightarrow \sum_{n=0}^{\infty} (a e^{-j\omega})^n$$

$$\left[1 + a + a^2 + \dots = \frac{1}{1-a}\right]$$

$$= 1 + a e^{-j\omega} + (a e^{-j\omega})^2 + \dots \infty$$

$$X(\omega) = \frac{1}{1 - a e^{-j\omega}}$$

(2) Find DTFT of $x(n) = u(n)$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} u(n)\, e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} e^{-j\omega n}$$

$$= 1 + e^{-j\omega} + (e^{-j\omega})^2 + \dots$$

$$X(\omega) = \frac{1}{1 - e^{-j\omega}}$$

(3) Find the Fourier transform of $x(n) = \delta(n)$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \delta(n)\, e^{-j\omega \cancel{n}^{1}}$$

$$\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

$$X(\omega) = 1$$

4) Find the DTFT of $x(n) = \delta(n-n_0)$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \delta(n-n_0)\, e^{-j\omega n}$$

$n-n_0 = m$, $n = m+n_0$

$$= \sum_{n=-\infty}^{\infty} \delta(m)\, e^{-j\omega(m+n_0)}$$

$$= \sum_{n=-\infty}^{\infty} \delta(m)\, e^{-j\omega m}\, e^{-j\omega n_0}$$

$$= \underbrace{\sum_{n=-\infty}^{\infty} \delta(m)\, e^{-j\omega m}}\; e^{-j\omega n_0}$$

$$X(\omega) = e^{-j\omega n_0}$$

(5) Find the DTFT of discrete time rectangular pulse of amplitude A and length L.

$$x(n) = \begin{cases} A, & 0 \le n \le L-1 \\ 0, & \text{otherwise} \end{cases}$$

$$\sum_{k=N_1}^{N_2} a^k = \frac{a^{N_1} - a^{N_2+1}}{1-a}$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$= \sum_{n=0}^{L-1} A\, e^{-j\omega n}$$

$$= A\left[\frac{(e^{-j\omega})^0 - e^{-j\omega L - \cancel{1} + \cancel{1}}}{1-e^{-j\omega}}\right]$$

$$= A\left[\frac{1-e^{-j\omega L}}{1-e^{-j\omega}}\right]$$

$$= A\left[\frac{e^{j\omega L/2}\,e^{-j\omega L/2} - e^{-j\omega L/2}\,e^{-j\omega L/2}}{e^{j\omega/2}\,e^{-j\omega/2} - e^{-j\omega/2}\,e^{-j\omega/2}}\right]$$

$$= A\left[\frac{e^{-j\omega L/2}\left(2j\sin \omega L/2\right)}{e^{-j\omega/2}\left(2j\sin \omega/2\right)}\right]$$

$$= A\left[e^{-j\omega L/2}\,e^{j\omega/2}\left[\frac{\sin \omega L/2}{\sin \omega/2}\right]\right]$$

$$= A\left[e^{-j\omega/2(L-1)}\left[\frac{\sin \omega L/2}{\sin \omega/2}\right]\right]$$