Properties of z-transform :-

1) Linearity :-

If $x_1(n) \overset{ZT}{\longleftrightarrow} X_1(z)$, $x_2(n) \overset{ZT}{\longleftrightarrow} X_2(z)$ then

$$a x_1(n) + b x_2(n) \overset{ZT}{\longleftrightarrow} a X_1(z) + b X_2(z)$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} [a x_1(n) + b x_2(n)] z^{-n}$$

$$= a \sum_{n=-\infty}^{\infty} x_1(n) z^{-n} + b \sum_{n=-\infty}^{\infty} x_2(n) z^{-n}$$

$$= a X_1(z) + b X_2(z)$$

2) Time shifting (or) Translation property :-

If $x(n) \overset{ZT}{\longleftrightarrow} X(z)$ then $x(n-k) \overset{ZT}{\longleftrightarrow} z^{-k} X(z)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n-k) z^{-n}$$

$n-k = m$, $n = m+k$

$$= \sum_{n=-\infty}^{\infty} x(m) z^{-(m+k)}$$

$$= \underbrace{\sum_{n=-\infty}^{\infty} x(m) z^{-m}} \cdot z^{-k} \Rightarrow z^{-k} X(z)$$

3) Scaling in z-domain :-

If $x(n) \leftrightarrow X(z)$ then $a^n x(n) \leftrightarrow X(a^{-1}z)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a^n x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) (a^{-1}z)^{-n}$$

$$= X(a^{-1}z)$$

4) Time Reversal property :-

If $x(n) \leftrightarrow X(z)$ then $x(-n) \leftrightarrow X(z^{-1})$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(-n) z^{-n}$$

$m = -n$ $$= \sum_{n=-\infty}^{\infty} x(m) [z^{-1}]^{m}$$

$$= X(z^{-1})$$

5) convolution :-

If $x(n) \leftrightarrow X(z)$ then $x(n) * h(n) \leftrightarrow X(z) H(z)$

Definition :-

$$x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k)$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} [x(n) * h(n)] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x(k) h(n-k) z^{-n+k-k}$$

$$= \sum_{k=-\infty}^{\infty} x(k) z^{-k} \sum_{n=-\infty}^{\infty} h(n-k) z^{-(n-k)}$$

$n-k=m$

$$= \sum_{k=-\infty}^{\infty} x(k) z^{-k} \sum_{n=-\infty}^{\infty} h(m) z^{-m}$$

$$= X(z) \cdot H(z)$$

6) Time Expansion :-

If $x(n) \leftrightarrow X(z)$ then $x_k(n) \leftrightarrow x(n/k)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n/k) z^{-n}$$

$n/k = m$, $n = km$

$$= \sum_{n=-\infty}^{\infty} x(m) z^{-km}$$

$$= \sum_{n=-\infty}^{\infty} x(m) (z^k)^{-m}$$

$$= X(z^k)$$

7) conjugation :-

If $x(n) \leftrightarrow X(z)$ then $x^*(n) \leftrightarrow X^*(z^*)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x^*(n) z^{-n}$$

$$= \left[\sum_{n=-\infty}^{\infty} x(n) (z^*)^{-n}\right]^* \Rightarrow [X(z^*)]^*$$

$$= X^*(z)$$

8) Multiplication by n (or) Differentiation in Z-Domain :-

If $x(n) \leftrightarrow X(z)$ then $n\,x(n) \xleftrightarrow{ZT} -z \, d/dz \, X(z)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} n\,x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} n\,x(n) z^{-n+1-1}$$

$$= z \sum_{n=-\infty}^{\infty} n\,x(n) z^{-(n+1)}$$

Differentiating :

$$d/dz \; X(z) = z \sum_{n=-\infty}^{\infty} n\,x(n) \frac{z^{-(n+1)}}{-(n+1)}$$

$$= z \sum_{n=-\infty}^{\infty} x(n) \left[n z^{-(n+1)} \right]$$

$$= z \sum_{n=-\infty}^{\infty} x(n) \; -d/dz \; z^{-n}$$

$$= -z \; d/dz \sum_{n=-\infty}^{\infty} x(n) z^{-n} \qquad \frac{d}{dz} z^{-n} = -n z^{-(n+1)}$$

$$= -z \; d/dz \; X(z)$$

9) Parseval's Relation :-

Let us consider two complex valued sequences $x_1(n)$ & $x_2(n)$. parseval's relation states that

$$\sum_{n=-\infty}^{\infty} x_1(n) \; x_2^*(n) = \frac{1}{2\pi j} \oint_c X_1(V) \; X_2^*\left(\frac{1}{V^*}\right) V^{-1} dV$$

$$\therefore Z\left[x_1(n) \; x_2^*(n) \right] = \sum_{n=-\infty}^{\infty} x_1(n) \; x_2^*(n) \; z^{-n}$$

By using complex convolution Theorem

$$= \frac{1}{2\pi j} \oint_C x_1(V)\, x_2^*\left(\frac{z^*}{V^*}\right) V^{-1}\, dV$$

Sub $z^* = 1$

$$= \frac{1}{2\pi j} \oint_C x_1(V)\, x_2^*\left(\frac{1}{V^*}\right) V^{-1}\, dV$$