DEPARTMENT OF MATHEMATICS

UNIT III

Envelope: A curve which touches each member of a family of curves is called the envelope of that family curves.

Problems:

(1) Find the envelope of the following:

(i) $y = mx + \frac{1}{m}$ (ii) $y = mx + \sqrt{a^2m^2+b^2}$

(iii) $y = mx + \frac{3}{2m}$ (iv) $(x-\alpha)^2 + (y-\alpha)^2 = 2\alpha$

Solution:

(i) $y = mx + \frac{1}{m}$

$y = \frac{m^2x+1}{m}$

$my = m^2x+1$

$m^2x - my + 1 = 0$, which is quadratic in 'm'.

$A = x, \quad B = -y, \quad C = 1$

Envelope; $B^2 - 4AC = 0 \Rightarrow (-y)^2 - 4(x)(1) = 0$

$y^2 - 4x = 0$

$\boxed{y^2 = 4x}$

(ii) $y = mx + \sqrt{a^2m^2+b^2}$

$y - mx = \sqrt{a^2m^2+b^2}$

Squaring, $(y-mx)^2 = a^2m^2+b^2$

$$y^2+m^2x^2-2ymx = a^2m^2+b^2$$

$$y^2+m^2x^2-2ymx-a^2m^2-b^2=0$$

$$m^2(x^2-a^2)-2ymx+y^2-b^2=0$$

Here $A = x^2-a^2$, $B=-2yx$, $C=y^2-b^2$

Envelope: $B^2-4AC=0$

$$(-2yx)^2-4(x^2-a^2)(y^2-b^2)=0$$

$$4y^2x^2-4\left[x^2y^2-x^2b^2-a^2y^2+a^2b^2\right]=0$$

$$\cancel{y^2x^2}-\cancel{x^2y^2}+x^2b^2+a^2y^2-a^2b^2=0$$

$$x^2b^2+a^2y^2=a^2b^2$$

$\div$ by $a^2b^2 \Rightarrow \boxed{\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1}$

(iii) $y = mx+\dfrac{3}{2m}$

$$y=\frac{2m^2x+3}{2m}$$

$$2my = 2m^2x+3$$

$$2m^2x-2my+3=0$$

$A=2x$, $B=-2y$, $C=3$

$B^2-4AC=0$

$\Rightarrow 4y^2-4(2x)(3)=0$

$$y^2-6x=0$$

$$\boxed{y^2=6x}$$

(iv) $(x-\alpha)^2+(y-\alpha)^2=2\alpha$

$$x^2+\alpha^2-2\alpha x+y^2-2\alpha y+y^2=2\alpha$$

$$2\alpha^2-2\alpha(x+y+1)+(x^2+y^2)=0$$

$\Rightarrow A=2$, $B=-2(x+y+1)$, $C=x^2+y^2$

$B^2-4AC=0$

$\Rightarrow 4(x+y+1)^2-4(2)(x^2+y^2)=0$

$\Rightarrow \boxed{(x+y+1)^2=2(x^2+y^2)}$

Envelope of two parameters:

(1) Find the envelope of the family of straight lines $\frac{x}{a}+\frac{y}{b}=1$ where a and b are connected by the relation (i) $a+b=c$ (ii) $ab=c^2$ where c is a constant.

Solution:

(i)
$$\frac{x}{a}+\frac{y}{b}=1 \rightarrow (1)$$

$$a+b=c \rightarrow (2)$$

$$\Rightarrow b=c-a$$

subs 'b' in (1),

$$\frac{x}{a}+\frac{y}{c-a}=1$$

$$x(c-a)+ya=a(c-a)$$

$$xc-ax+ya=ac-a^2$$

$$xc-ax+ya-ac+a^2=0$$

$$a^2+a(y-x-c)+cx=0.$$

Here $A=1$, $B=y-x-c$, $C=cx$

Envelope: $B^2-4AC=0$

$$(y-x-c)^2-4(1)(cx)=0$$

$$(y-x-c)^2-4cx=0$$

(ii) $\frac{x}{a} + \frac{y}{b} = 1 \rightarrow (1)$

$ab = c^2 \rightarrow (2)$

$\Rightarrow b = c^2/a$

Subs 'b' in (1),

$\frac{x}{a} + \frac{y}{c^2/a} = 1$

$\frac{x}{a} + \frac{ay}{c^2} = 1$

$c^2 x + a^2 y = ac^2$

$a^2 y - ac^2 + c^2 x = 0$

$A = y, \; B = -c^2, \; C = c^2 x$

$B^2 - 4AC = 0$

$(-c^2)^2 - 4(y)(c^2 x) = 0$

$c^4 - 4xyc^2 = 0$

$\div$ by $c^2 \Rightarrow c^2 - 4xy = 0$

$\boxed{4xy = c^2}$