UNIT V

Triple integration :

① Evaluate $\int_0^3 \int_0^2 \int_0^1 (x+y+z)\, dz\, dy\, dx$

$$= \int_0^3 \int_0^2 \left[xz + yz + \frac{z^2}{2} \right]_0^1 dy\, dx$$

$$= \int_0^3 \int_0^2 \left[x(1-0) + y(1-0) + \frac{1}{2}(1-0) \right] dy\, dx$$

$$= \int_0^3 \int_0^2 \left[x + y + \frac{1}{2} \right] dy\, dx$$

$$= \int_0^3 \left[xy + \frac{y^2}{2} + \frac{1}{2} y \right]_0^2 dx$$

$$= \int_0^3 \left[x(2-0) + \frac{1}{2}(2^2 - 0) + \frac{1}{2}(2-0) \right] dx$$

$$= \int_0^3 \left[2x + 2 + 1 \right] dx$$

$$= \int_0^3 (2x+3)\, dx$$

$$= \left[\cancel{2} \cdot \frac{x^2}{\cancel{2}} + 3x \right]_0^3$$

$$= (3^2 - 0 + 3(3-0))$$

$$= 9 + 9$$

$$I = 18$$

(2) Evaluate $\int_0^1 \int_0^2 \int_0^3 xyz \, dx \, dy \, dz$

$$= \int_0^1 \int_0^2 \left[\int_0^3 x \, dx \right] yz \, dy \, dz$$

$$= \int_0^1 \int_0^2 \left[\frac{x^2}{2} \right]_0^3 yz \, dy \, dz$$

$$= \int_0^1 \int_0^2 \left[\frac{1}{2}(3^2 - 0) \right] yz \, dy \, dz$$

$$= \int_0^1 \int_0^2 \frac{9}{2} yz \, dy \, dz$$

$$= \frac{9}{2} \int_0^1 \left[\int_0^2 y \, dy \right] z \, dz$$

$$= \frac{9}{2} \int_0^1 \left[\frac{y^2}{2} \right]_0^2 z \, dz$$

$$= \frac{9}{2} \int_0^1 \frac{1}{2}(2^2 - 0) z \, dz$$

$$= \frac{9}{2} \int_0^1 \frac{4}{2} z \, dz$$

$$= 9 \left[\frac{z^2}{2} \right]_0^1$$

$$= \frac{9}{2}[1 - 0] = \frac{9}{2}$$

$$\boxed{I = \frac{9}{2}}$$

(3) Evaluate $\int_0^a \int_0^b \int_0^c e^{x+y+z}\, dx\, dy\, dz$.

$$= \int_0^a \int_0^b \int_0^c e^x e^y e^z\, dx\, dy\, dz$$

$$= \int_0^a \int_0^b \left[\int_0^c e^x\, dx\right] e^y e^z\, dy\, dz$$

$$= \int_0^a \int_0^b \left[e^x\right]_0^c e^y e^z\, dy\, dz$$

$$= \int_0^a \int_0^b (e^c - e^0) e^y e^z\, dy\, dz$$

$$= (e^c - 1) \int_0^a \left[\int_0^b e^y\, dy\right] e^z\, dz$$

$$= (e^c - 1) \int_0^a \left[e^y\right]_0^b e^z\, dz$$

$$= (e^c - 1) \int_0^a \left[e^b - e^0\right] e^z\, dz$$

$$= (e^c - 1)(e^b - 1) \int_0^a e^z\, dz$$

$$= (e^c - 1)(e^b - 1) \left[e^z\right]_0^a$$

$$= (e^c - 1)(e^b - 1)(e^a - e^0)$$

$$\boxed{I = (e^c - 1)(e^b - 1)(e^a - 1)}$$