UNIT IV

JACOBIANS

If $u = f(x,y)$, $v = g(x,y)$ are two functions then Jacobian of u and v w.r.t x, y is denoted by $\frac{\partial(u,v)}{\partial(x,y)}$ (or) J and is defined by,

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

For three variables (u, v, w) which are functions of (x, y, z) then Jacobian is,

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

PROPERTIES OF JACOBIANS :

* If u and v are functions of x and y and x and y are functions of r and s, then

$$\frac{\partial(u,v)}{\partial(r,s)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,s)}$$

* u and v are functions of x and y, then

$$\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = 1 \Rightarrow JJ' = 1$$

* Functional dependence

If $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$, then u, v and w are functionally dependent.

PROBLEMS :

1. Find the Jacobian of the following transformati

(i) $u = \frac{2x-y}{2}$, $v = y/2$. Find $\frac{\partial(u,v)}{\partial(x,y)}$

(ii) $u = xyz$, $v = xy+yz+zx$, $w = x+y+z$. Find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

(iii) $u = 2xy$, $v = x^2/y$

(iv) $u = y^2/x$, $v = x^2/y$ } Find $\frac{\partial(u,v)}{\partial(x,y)}$

(V) $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ Find $\frac{\partial(x,y,z)}{\partial(r,\theta,z)}$

(vi) $x = u(1+v)$, $y = v(1+u)$. Find $\frac{\partial(x,y)}{\partial(u,v)}$

(vii) $x = \frac{u^2}{v}$, $y = \frac{v^2}{w}$, $z = \frac{w^2}{u}$. Find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$

(viii) $u = e^x \cos y$, $v = e^x \sin y$. Find $\frac{\partial(u,v)}{\partial(x,y)}$

(ix) $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$. Find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

Soln:

(i) $J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & -1/2 \\ 0 & 1/2 \end{vmatrix} = 1/2$

(ii) $J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \frac{\partial(u,v,w)}{\partial(x,y,z)}$

$= \begin{vmatrix} yz & xz & xy \\ y+z & x+z & x+y \\ 1 & 1 & 1 \end{vmatrix}$

$= (x-y)(y-z)(z-x)$

(iii) $J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

$= \begin{vmatrix} 2y & 2x \\ \frac{2x}{y} & \frac{-x^2}{y^2} \end{vmatrix}$

$= \frac{-6x^2}{y}$

(iv) $J = \dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -y^2/x^2 & 2y/x \\ 2x/y & -x^2/y^2 \end{vmatrix} = -3$

$\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{J} = -1/3$

(v) $J = \dfrac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$

$= \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$= r$

(vi) $J = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix}$

$= 1+u+v$

(vii) $J = \dfrac{\partial(x,y,z)}{\partial(u,v,w)}$

$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} \frac{2u}{v} & \frac{-u^2}{v^2} & 0 \\ 0 & \frac{2v}{w} & \frac{-v^2}{w^2} \\ \frac{-w^2}{u^2} & 0 & \frac{2w}{u} \end{vmatrix}$

$= 7$

(viii) $J = \dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} e^{x}\cos y & -e^{x}\sin y \\ e^{x}\sin y & e^{x}\cos y \end{vmatrix}$

$= e^{2x}$

(ix) $J = \begin{vmatrix} \dfrac{-yz}{x^2} & \dfrac{z}{x} & \dfrac{y}{x} \\ z/y & -zx/y^2 & \dfrac{x}{y} \\ y/z & x/z & -xy/z^2 \end{vmatrix} = 4$

② If $u = 2xy$, $v = x^2 - y^2$ and $x = r\cos\theta$, $y = r\sin\theta$

Evaluate $\dfrac{\partial(r,\theta)}{\partial(u,v)}$.

Soln:

u, v

x, y

r, θ

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)}$$

$$= \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} 2y & 2x \\ 2x & -2y \end{vmatrix} \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= -4r^3$$

$$\frac{\partial(r,\theta)}{\partial(u,v)} = \frac{1}{\dfrac{\partial(u,v)}{\partial(r,\theta)}} = \frac{-1}{4r^3}$$

③ If $x+y+z = u$, $y+z = uv$, $z = uvw$ find

$\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$.