UNIT V

Volume of triple integrals:

(1) Find the volume of the sphere $x^2+y^2+z^2=a^2$ without transformation.

Soln:

Volume = 8 × Volume in the 1^{st} octant.

z limits:

$$x^2+y^2+z^2=a^2$$

$$z^2=a^2-x^2-y^2$$

$$z=\pm\sqrt{a^2-x^2-y^2}$$

z varies from 0 to $\sqrt{a^2-x^2-y^2}$

y limits: Put $z=0$ in $x^2+y^2+z^2=a^2$

$$x^2+y^2=a^2$$

$$y^2=a^2-x^2$$

$$y=\pm\sqrt{a^2-x^2}$$

y varies from 0 to $\sqrt{a^2-x^2}$

x limit: Put $y=0$, $z=0$ in $x^2+y^2+z^2=a^2$

$$x^2=a^2$$

$$x=\pm a$$

x varies from 0 to a.

$$V = 8 \iiint dz\, dy\, dx$$

$$= 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dz\, dy\, dx$$

$$= 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} [z]_0^{\sqrt{a^2-x^2-y^2}} dy\, dx$$

$$= 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\, dy\, dx$$

$$= 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{(\sqrt{a^2-x^2})^2-y^2}\, dy\, dx$$

$$= 8 \int_0^a \left[\frac{y}{2}\sqrt{a^2-x^2-y^2} + \frac{a^2-x^2}{2}\sin^{-1}\left(\frac{y}{\sqrt{a^2-x^2}}\right)\right]_0^{\sqrt{a^2-x^2}} dx$$

Formula :

$$\left[\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right.$$

Here $a = \sqrt{a^2-x^2}$, $x = y$]

$$= 8 \int_0^a \left[0 + \frac{a^2-x^2}{2}\sin^{-1}\left(\frac{\sqrt{a^2-x^2}}{\sqrt{a^2-x^2}}\right) - 0\right] dx$$

$$= 8 \int_0^a \left(\frac{a^2-x^2}{2}\right)\sin^{-1}(1)\, dx$$

$$= 8 \int_0^a \left(\frac{a^2-x^2}{2}\right)\frac{\pi}{2}\, dx$$

$$V = \frac{8\pi}{2\times 2} \int_0^a (a^2 - x^2)\,dx$$

$$= 2\pi \left[a^2 x - \frac{x^3}{3} \right]_0^a$$

$$= 2\pi \left[a^2 . a - \frac{a^3}{3} \right]$$

$$= 2\pi \left[a^3 - \frac{a^3}{3} \right]$$

$$= 2\pi \left[\frac{3a^3 - a^3}{3} \right]$$

$$V = 2\pi \left(\frac{2a^3}{3} \right)$$

$$V = \frac{4\pi a^3}{3} \text{ cubic units}$$