



DEPARTMENT OF MATHEMATICS

UNIT III

DIFFERENTIAL CALCULUS

Evolute :

The locus of the centre of curvature of the given curve is called the evolute of the curve.

Working procedure to find the evolute :

1. Write the parametric form of the given curve.
2. Find centre of curvature (\bar{x}, \bar{y}) .
3. Eliminate the parameter from \bar{x} and \bar{y}
4. Taking locus of the above equation, we get the required evolute.

Curve	Cartesian equation	Parametric equation
Parabola	① $y^2 = 4ax$ ② $x^2 = 4ay$	① $x = at^2, y = 2at$ ② $x = 2at, y = at^2$
Ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$x = a \cos \theta, y = b \sin \theta$
Hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$x = a \sec \theta, y = b \tan \theta$
Rectangular hyperbola	$xy = c^2$	$x = ct, y = \frac{c}{t}$
Astroid	$x^{2/3} + y^{2/3} = a^{2/3}$	$x = a \cos^3 \theta, y = a \sin^3 \theta$

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Problems :

- ① Find the equation of the evolute of the parabola $y^2 = 4ax$.

Soln:

The parametric equations of the parabola $y^2 = 4ax$ are $x = at^2$, $y = 2at$.

$$\frac{dx}{dt} = 2at, \quad \frac{dy}{dt} = 2a$$

$$y_1 = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2a}{2at} = \frac{1}{t}$$

$$\begin{aligned} y_2 = \frac{d^2y}{dx^2} &= \frac{d}{dx} \left(\frac{dy}{dx} \right) \\ &= \frac{d}{dx} \left(\frac{1}{t} \right) \\ &= \frac{d}{dx} (t^{-1}) \\ &= \frac{-1}{t^2} \cdot \frac{dt}{dx} \\ &= \frac{-1}{t^2} \cdot \frac{1}{dx/dt} \\ &= \frac{-1}{t^2} \cdot \frac{1}{2at} \end{aligned}$$

$$y_2 = \frac{-1}{2at^3}$$

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$$\bar{x} = x - \frac{y_1}{y_2} (1 + y_1^2)$$

$$= at^2 - \frac{1/t}{-1/2at^3} \left(1 + \frac{1}{t^2}\right)$$

$$= at^2 + \frac{1}{t} \cdot 2at^3 \left(1 + \frac{1}{t^2}\right)$$

$$= at^2 + 2at^2 \left(1 + \frac{1}{t^2}\right)$$

$$= at^2 + 2at^2 + 2at^2 \cdot \frac{1}{t^2}$$

$$= at^2 + 2at^2 + 2a$$

$$\bar{x} = 3at^2 + 2a \Rightarrow \textcircled{1} \quad t^2 = \frac{\bar{x} - 2a}{3a} \rightarrow \textcircled{1}$$

$$\bar{y} = y + \frac{1}{y_2} (1 + y_1^2)$$

$$= 2at + \frac{1}{-1/2at^3} \left(1 + \frac{1}{t^2}\right)$$

$$= 2at - 2at^3 \left(1 + \frac{1}{t^2}\right)$$

$$= 2at - 2at^3 - 2at^3 \cdot \frac{1}{t^2}$$

$$= 2at - 2at^3 - 2at$$

$$\bar{y} = -2at^3 \Rightarrow t^3 = -\frac{\bar{y}}{2a} \rightarrow \textcircled{2}$$

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Taking cube of (1) & Squaring (2) we get,

$$(t^2)^3 = \left(\frac{\bar{x} - 2a}{2a}\right)^3$$

$$t^6 = \frac{(\bar{x} - 2a)^3}{27a^3} \rightarrow (3)$$

$$(t^3)^2 = \left(\frac{-\bar{y}}{2a}\right)^2$$

$$t^6 = \frac{\bar{y}^2}{4a^2} \rightarrow (4)$$

From (3) & (4),

$$\frac{(\bar{x} - 2a)^3}{27a^3} = \frac{\bar{y}^2}{4a^2}$$

$$\frac{(\bar{x} - 2a)^3}{27a} = \frac{\bar{y}^2}{4}$$

$$4(\bar{x} - 2a)^3 = 27a\bar{y}^2$$

Locus of (\bar{x}, \bar{y}) is,

$$4(x - 2a)^3 = 27ay^2$$

which is the evolute of the parabola $y^2 = 4ax$.

