UNIT IV

(6) We can prove the following Statements:

(i) $u_{xy} = u_{yx}$ when $u = \tan^{-1}\left(\frac{x}{y}\right)$

(ii) $u = (x^2+y^2+z^2)^{-1/2}$, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

(iii) $u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{y}{x}\right)$, $\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2-y^2}{x^2+y^2}$

EULER'S THEOREM FOR HOMOGENEOUS FUNCTIONS

HOMOGENEOUS FUNCTION:

A homogeneous function of degree n in x and y is $f(x,y) = x^n f\left(\frac{y}{x}\right)$ or a function $f(x,y)$ is said to be homogeneous function in x and y of degree n if $f(tx, ty) = t^n f(x,y)$.

Euler's theorem for Homogeneous function

If $u = f(x,y)$ is a homogeneous function of degree n then $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

PROBLEMS:

(1) Verify Euler's theorem for $u = x^3 \sin\left(\frac{y}{x}\right)$

Soln:

$u = x^3 \sin\left(\frac{y}{x}\right)$

Let $u = f(x,y) = x^3 \sin\left(\frac{y}{x}\right)$

$$f(tx, ty) = t^3 x^3 \sin\left(\frac{ty}{tx}\right)$$

$$= t^3 x^3 \sin(y/x)$$

$$= t^3 f(x,y)$$

$\therefore$ u is a homogeneous function of degree 3.

To prove: $x \frac{\partial u}{\partial x} + y \left(\frac{\partial u}{\partial y}\right) = 3u$

$$\frac{\partial u}{\partial x} = 3x^2 \sin\left(\frac{y}{x}\right) - xy \cos\left(\frac{y}{x}\right)$$

$$\frac{\partial u}{\partial y} = x^2 \cos\left(\frac{y}{x}\right)$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 3x^3 \sin\left(\frac{y}{x}\right) = 3u$$

Hence proved.

(2) Verify Euler's theorem for the following functions:

(i) $u = (x^2 + y^2 + xy)^{-1}$

(ii) $u = ax^2 + 2hxy + by^2$

(iii) $u = x \log\left(\frac{y}{x}\right)$

Soln:

(i) u is a homogeneous function of degree -2

$$\frac{\partial u}{\partial x} = -(2x+y)(x^2+y^2+xy)^{-2}$$

$$\frac{\partial u}{\partial y} = -(2y+x)(x^2+y^2+xy)^{-2}$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = -2u$$

(ii) u is homogeneous function of degree 2

$$\frac{\partial u}{\partial x} = 2ax + 2hy$$

$$\frac{\partial u}{\partial y} = 2hx + 2by$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2u$$

(iii) u is a homogeneous function of degree 1

$$\frac{\partial u}{\partial x} = -1 + \log\left(\frac{y}{x}\right)$$

$$\frac{\partial u}{\partial y} = \frac{x}{y}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u.$$

(3) Verify Euler's theorem for $u = \frac{1}{\sqrt{x^2+y^2}}$

Soln:

$$f(x,y) = u = \frac{1}{\sqrt{x^2+y^2}}$$

$$f(tx,ty) = \frac{1}{t\sqrt{x^2+y^2}} = t^{-1} f(x,y)$$

u is homogeneous function of degree -1.

$$\frac{\partial u}{\partial x} = -x\,(x^2+y^2)^{-3/2}$$

$$\frac{\partial u}{\partial y} = -y\,(x^2+y^2)^{-3/2}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -(x^2+y^2)^{-1/2} = -u.$$

(4) Prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$ using Euler's theorem if $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$

Soln:

$$u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$$

$$\tan u = \frac{x^3+y^3}{x-y}$$

$$f(x,y) = \tan u = \frac{x^3+y^3}{x-y}$$

$\therefore$ tan u is a homogeneous function of degree 2.

By Euler's theorem,

$$x \frac{\partial}{\partial x}(\tan u) + y \frac{\partial}{\partial y}(\tan u) = 2 \tan u$$

$$x \sec^2 u \frac{\partial u}{\partial x} + y \sec^2 u \frac{\partial u}{\partial y} = 2 \tan u$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{2 \tan u}{\sec^2 u}$$

$$= 2 \frac{\sin u}{\cos u} . \cos^2 u$$

$$\therefore x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \sin 2u$$

(5) Using Euler's theorem, prove that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0$ if $u = \sin^{-1}\left(\frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}\right)$

Soln:

$$\sin u = \frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}$$

Sin u is a homogeneous function of degree 0.

By Euler's theorem,

$$x \frac{\partial}{\partial x} \sin u + y \frac{\partial}{\partial y} \sin u = 0$$

$$x \cos u \frac{\partial u}{\partial x} + y \cos u \frac{\partial u}{\partial y} = 0$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0$$

(6) Using Euler's theorem, prove that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 3u$ if $u = xy^2 \sin\left(\frac{x}{y}\right)$

Soln: