## UNIT III

**Envelope:** A curve which touches each member of a family of curves is called the envelope of that family curves.

**Problems:**

(1) Find the envelope of the following:

(i) $y = mx + \frac{1}{m}$ (ii) $y = mx + \sqrt{a^2m^2+b^2}$

(iii) $y = mx + \frac{3}{2m}$ (iv) $(x-\alpha)^2 + (y-\alpha)^2 = 2\alpha$

**Solution:**

(i) $y = mx + \frac{1}{m}$

$y = \frac{m^2x+1}{m}$

$my = m^2x+1$

$m^2x - my + 1 = 0$, which is quadratic in 'm'.

$A = x,\ B = -y,\ C = 1$

**Envelope:** $B^2 - 4AC = 0 \Rightarrow (-y)^2 - 4(x)(1) = 0$

$y^2 - 4x = 0$

$\boxed{y^2 = 4x}$

(ii) $y = mx + \sqrt{a^2m^2+b^2}$

$y - mx = \sqrt{a^2m^2+b^2}$

Squaring, $(y-mx)^2 = a^2m^2+b^2$

$$y^2 + m^2x^2 - 2ymx = a^2m^2 + b^2$$

$$y^2 + m^2x^2 - 2ymx - a^2m^2 - b^2 = 0$$

$$m^2(x^2 - a^2) - 2ymx + y^2 - b^2 = 0$$

Here $A = x^2 - a^2$, $B = -2yx$, $C = y^2 - b^2$

<u>Envelope</u> : $B^2 - 4AC = 0$

$$(-2yx)^2 - 4(x^2 - a^2)(y^2 - b^2) = 0$$

$$4y^2x^2 - 4\left[x^2y^2 - x^2b^2 - a^2y^2 + a^2b^2\right] = 0$$

$$\cancel{y^2x^2} - \cancel{x^2y^2} + x^2b^2 + a^2y^2 - a^2b^2 = 0$$

$$x^2b^2 + a^2y^2 = a^2b^2$$

$\div$ by $a^2b^2 \Rightarrow \boxed{\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1}$

(iii) $y = mx + \dfrac{3}{2m}$

$$y = \frac{2m^2x + 3}{2m}$$

$$2my = 2m^2x + 3$$

$$2m^2x - 2my + 3 = 0$$

$A = 2x$, $B = -2y$, $C = 3$

$B^2 - 4AC = 0$

$$\Rightarrow 4y^2 - 4(2x)(3) = 0$$

$$y^2 - 6x = 0$$

$$\boxed{y^2 = 6x}$$

(iv) $(x - \alpha)^2 + (y - \alpha)^2 = 2\alpha$

$$x^2 + \alpha^2 - 2\alpha x + y^2 - 2\alpha y + y^2 = 2\alpha$$

$$2\alpha^2 - 2\alpha(x + y + 1) + (x^2 + y^2) = 0$$

$\Rightarrow A = 2$, $B = -2(x + y + 1)$

$C = x^2 + y^2$

$B^2 - 4AC = 0$

$$\Rightarrow 4(x + y + 1)^2 - 4(2)(x^2 + y^2) = 0$$

$$\Rightarrow \boxed{(x + y + 1)^2 = 2(x^2 + y^2)}$$

## Envelope of two parameters:

1. Find the envelope of the family of straight lines $\frac{x}{a}+\frac{y}{b}=1$ where a and b are connected by the relation (i) $a+b=c$ (ii) $ab=c^2$ where c is a constant.

Solution:

(i)
$$\frac{x}{a}+\frac{y}{b}=1 \rightarrow (1)$$

$$a+b=c \rightarrow (2)$$

$$\Rightarrow b=c-a$$

subs 'b' in (1),

$$\frac{x}{a}+\frac{y}{c-a}=1$$

$$x(c-a)+ya=a(c-a)$$

$$xc-ax+ya=ac-a^2$$

$$xc-ax+ya-ac+a^2=0$$

$$a^2+a(y-x-c)+cx=0.$$

Here $A=1$, $B=y-x-c$, $C=cx$

Envelope: $B^2-4AC=0$

$$(y-x-c)^2-4(1)(cx)=0$$

$$(y-x-c)^2-4cx=0$$

(ii) $\frac{x}{a} + \frac{y}{b} = 1 \rightarrow (1)$

$ab = c^2 \rightarrow (2)$

$\Rightarrow b = c^2/a$

Subs 'b' in (1),

$\frac{x}{a} + \frac{y}{c^2/a} = 1$

$\frac{x}{a} + \frac{ay}{c^2} = 1$

$c^2x + a^2y = ac^2$

$a^2y - ac^2 + c^2x = 0$

$A = y, \; B = -c^2, \; C = c^2x$

$B^2 - 4AC = 0$

$(-c^2)^2 - 4(y)(c^2x) = 0$

$c^4 - 4xyc^2 = 0$

$\div$ by $c^2 \Rightarrow c^2 - 4xy = 0$

$\boxed{4xy = c^2}$