Constrained maxima and minima

* Consider a function $f(x, y, z)$ subject to the constraind $g(x, y, z) = 0$

Equation:

* $u(x, y, z) = f(x, y, z) + \lambda g(x, y, z)$
* where λ is non - determined constant called lagranjain multiplier

Methods of finding maxima and minima by Lagranjain multiplier

* Find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ and equal to zero.
* Solve x, y, z
* The values of x, y, z either maximum (or) minimum
* Here λ is a parameter independent of (x, y, z)

Example:

Find the maximum value of $x.y.z$ subject to the constrained $x + y + z = a$

Soln: Given, x, y, z

$$x + y + z = a$$

$$x + y + z - a = 0.$$

$$f(x, y, z) = xyz \longrightarrow (1)$$

$$g(x, y, z) = x + y + z - a \longrightarrow (2).$$

Lagranjain formula,

$$u(x, y, z) = f(x, y, z) + \lambda g(x, y, z)$$

$$= xyz + \lambda(x+y+z-a)$$

$\frac{\partial u}{\partial x} = yz + \lambda$; $\frac{\partial u}{\partial y} = xz + \lambda$;

$\frac{\partial u}{\partial z} = xy + \lambda$

$yz + \lambda = 0$ | $xz + \lambda = 0$ | $xy + \lambda = 0$

$\boxed{yz = -\lambda}$ | $\boxed{xz = -\lambda}$ | $\boxed{xy = -\lambda}$

$$yz = xz = xy = -\lambda$$

$yz = xz$ $\quad$ $xz = xy$

$\boxed{y = x}$ $\quad$ $\boxed{z = y}$

$x = y = z$ sub in (2)

$$x + x + x - a = 0$$

$$3x - a = 0$$

$$x = a/3$$

$$y = a/3$$

$$z = a/3$$

To find maxima value,
sub x, y, z in eqn (1)

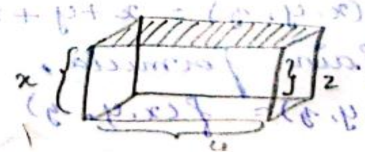
$$f(x,y,z) = xyz = (a/3)(a/3)(a/3) = \frac{a^3}{27}$$

(※) very very important

Example : 2

A rectangular box opened at the top is to have the volume of 32cc. Find the dimension of the box that it requires least materials for its construction.

Soln :



Let us consider x, y, z be length, breadth and height respectively

$S = xy + 2yz + 2xz$

Subjected to constrained

$xyz = 32cc$

$f(x,y,z) = xy + 2yz + 2xz \rightarrow ①$

$g(x,y,z) = xyz - 32 \rightarrow ②$.

Lagrangian multiplier

$u(x,y,z) = f(x,y,z) + \lambda g(x,y,z)$

$u(x,y,z) = xy + 2yz + 2xz + \lambda(xyz - 32) \rightarrow ③$

Differentiate ③ with respect to x, y, z

$\frac{\partial u}{\partial x} = y + 2z + \lambda yz$; $\frac{\partial u}{\partial y} = x + 2z + \lambda xz$;

$\frac{\partial u}{\partial x} = 0$; $\frac{\partial u}{\partial y} = 0$.

$\frac{\partial u}{\partial z} = 2y + 2x + \lambda xy$

$\frac{\partial u}{\partial z} = 0$

$y + 2z + \lambda yz = 0$	$x + 2z + \lambda xz = 0$	$2y + 2x + \lambda xy = 0$
$y + 2z = -\lambda yz$	$x + 2z = -\lambda xz$	$2y + 2x = -\lambda xy$
Multiply by x	⊗ by y	⊗ by z
$xy + 2xz = -\lambda xyz$	$xy + 2yz = -\lambda xyz$	$2yz + 2xz = -\lambda xyz$

$x + y + 2xz = xy + 2yz = 2yz + 2xz = -\lambda xyz$

$xy + 2xz = xy - 2yz$

$2xz = 2yz$

$\boxed{x = y}$

$$xy + 2yz = 2yz + 2xz$$
$$xy = 2xz$$
$$y = 2z$$

$x = y = 2z$ sub this in eqn ⑧

$$x = y \quad \Big| \quad x = 2z, \quad z = \frac{x}{2}$$

$$xyz - 32 = 0$$
$$x(x)\left(\frac{x}{2}\right) - 32 = 0$$
$$\frac{x^3}{2} - 32 = 0$$
$$\frac{x^3 - 64}{2} = 0$$
$$x^3 - 64 = 0$$
$$x^3 = 64$$
$$x^3 = 4^3$$
$$\boxed{x = 4}$$
$$\boxed{y = 4}$$
$$\boxed{z = 2}$$