Unit - 4

Functions of several variables

Partial differentiation :

* Let $u = f(x, y)$ be a functions of to independent variables, then differentiate u with respect to x keeping y as constant is known as the partial differential coefficient of u with respect to x.

Notes :

* $\frac{\partial u}{\partial x}$ - mean differentiate u with respect to x keeping y as constant.

* $\frac{\partial u}{\partial y}$ - means differentiate u with respect to y keeping x as constant

* $\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$

* $\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$

* $\frac{\partial^2 u}{\partial y \cdot \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$

* $\frac{\partial^2 u}{\partial x \cdot \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$

* $\frac{\partial^2 u}{\partial x \cdot \partial y} = \frac{\partial^2 u}{\partial y \cdot \partial x}$

Example : 1

Find the first and second derivative of $u = x^3 + y^3 - 3axy$

Soln:

$$\frac{\partial u}{\partial x} = 3x^2 - 3ay$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(3x^2 - 3ay)$$

$u = xe^y + ye^x$

$$\frac{\partial^2 u}{\partial x^2} = 6x$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}(3y^2 - 3ax) = -3a$$

$$\frac{\partial u}{\partial y} = 3y^2 - 3ax$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(3y^2 - 3ax)$$

$$\frac{\partial^2 u}{\partial y^2} = 6y$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}(3x^2 - 3ay - 3a)x$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}(3x^2 - 3ay - 3a) = -3a$$

(*) very important

Example: 2:

If $(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$ then show that $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2} = \frac{2}{r}$

Soln:

Differentiate ① with respect to 'x' partially

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2 \longrightarrow ①$$

$$2(x-a) = 2r \frac{\partial r}{\partial x}$$

$$\frac{\partial r}{\partial x} = \frac{x-a}{r}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial r}{\partial x}\right) \Rightarrow \frac{\partial}{\partial x}\left(\frac{x-a}{r}\right)$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{r(1) - (x-a)\frac{\partial r}{\partial x}}{r^2}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{r - (x-a)\left(\frac{x-a}{r}\right)}{r^2}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{r - \frac{(x-a)^2}{r}}{r^2} \quad r - \frac{(x-a)^2}{r}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{\frac{r^2-(x-a)^2}{r}}{r^2} \times \frac{1}{r^2} \quad \frac{r^2-(x-a)^2}{r}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{r^2-(x-a)^2}{r^3} \longrightarrow (2)$$

III^ly,

$$\frac{\partial r}{\partial y} = \frac{y-b}{r}$$

$$\frac{\partial^2 r}{\partial y^2} = \frac{r^2-(y-b)^2}{r^3} \longrightarrow (3)$$

III^ly,

$$\frac{\partial r}{\partial z} = \frac{z-c}{r}$$

$$\frac{\partial^2 r}{\partial z^2} = \frac{r^2-(z-c)^2}{r^3} \longrightarrow (4)$$

Now (2) + (3) + (4)

$$= \frac{r^2-(x-a)^2}{r^3} + \frac{r^2-(y-b)^2}{r^3} + \frac{r^2-(z-c)^2}{r^3}$$

$$= \frac{3r^2 - [(x-a)^2+(y-b)^2+(z-c)^2]}{r^3} \quad \frac{3r^2-r^2}{r^3} \quad \frac{2r^2}{r^3}\quad \frac{2}{r}$$

$$= \frac{3x^2 - r^2}{r^3}$$

$$= \frac{2r^2}{r^3}$$

$$= \frac{2}{r} \;//$$

Example: 3

Verify $U_{xy} = U_{yx}$ when $u = \tan^{-1}(x/y)$

Soln:

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

$$u = \tan^{-1}(x/y) \qquad \tan^{-1}x \to \frac{1}{1+x^2}$$

$$\frac{\partial u}{\partial x} = \frac{1}{1+(x/y)^2}\,(1/y)$$

$$= \frac{1}{1+x^2/y^2}\,(1/y)$$

$$= \frac{1}{\frac{y^2+x^2}{y^2}}\left(\frac{1}{y}\right) = \frac{y^2}{y^2+x^2} \times \frac{1}{y}$$

$$\left(\frac{y^2}{x^2+y^2}\right) = \frac{1}{y} \qquad = \frac{y}{y^2+x^2}$$

$$\frac{\partial u}{\partial x} = \frac{y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)$$

$$= \frac{(x^2+y^2)(1) - (y)(2y)}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}$$

$$= \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial x}{\partial y}\right)\left(x \cdot \frac{1}{y}\right)$$

$$\frac{\partial u}{\partial y} = \frac{1}{1+\left(\frac{x}{y}\right)^2}\left(\frac{-x}{y^2}\right)$$

$$= \frac{1}{\frac{y^2+x^2}{y^2}}\left(\frac{-x}{y^2}\right)$$

$$= \frac{y^2}{y^2+x^2}\left(\frac{-x}{y^2}\right)$$

$$= \frac{-x}{y^2+x^2}$$

$\frac{1}{y}$, y^{-1}, $-1y^{-1-1}$, $-1y^{-2}$, $-\frac{1}{y^2}$

$$\frac{\partial^2 x}{\partial x \cdot \partial y} = \frac{\partial}{\partial x}\left[\frac{-x}{x^2+y^2}\right]$$

$$= \frac{x^2+y^2(-1) - (-x)(2x)}{(x^2+y^2)^2}$$

$$= \frac{-x^2-y^2+2x^2}{(x^2+y^2)^2}$$

$$= \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\boxed{u_{xy} = u_{yx}}$$