SNS COLLEGE OF TECHNOLOGY

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Department of Biomedical Engineering

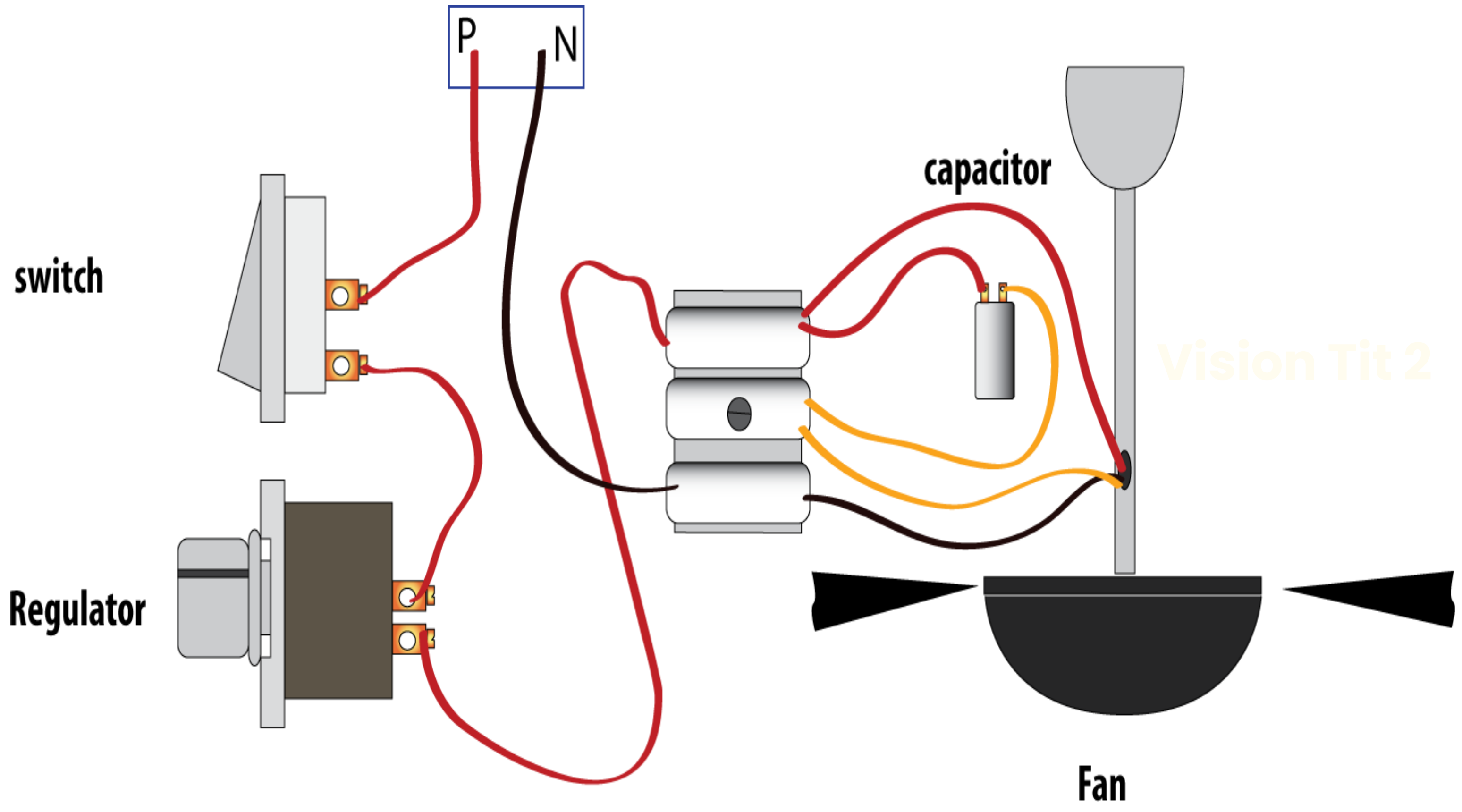
Course Name: 23BMT201 & Circuit Analysis

I Year : II Semester

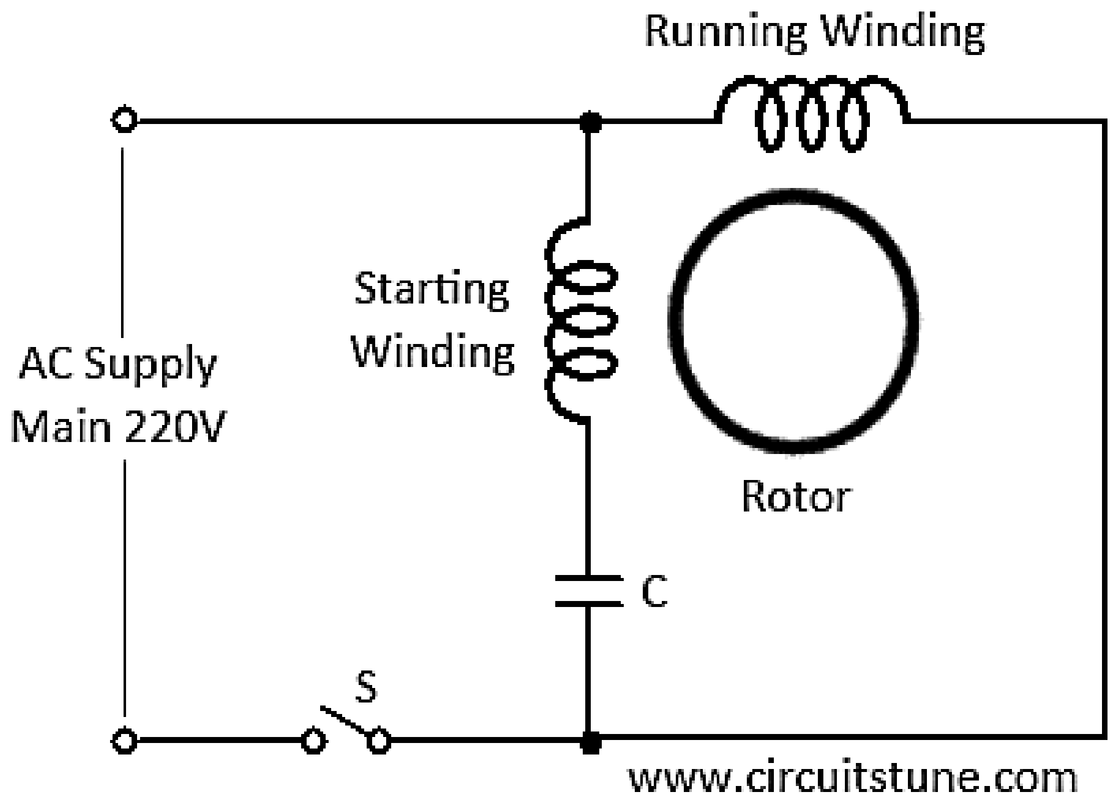
Unit IV –TRANSIENT RESPONSE FOR DC AND AC CIRCUITS

Topic : Step Response of RLC Circuits

Real Life Applications – Electric Fan

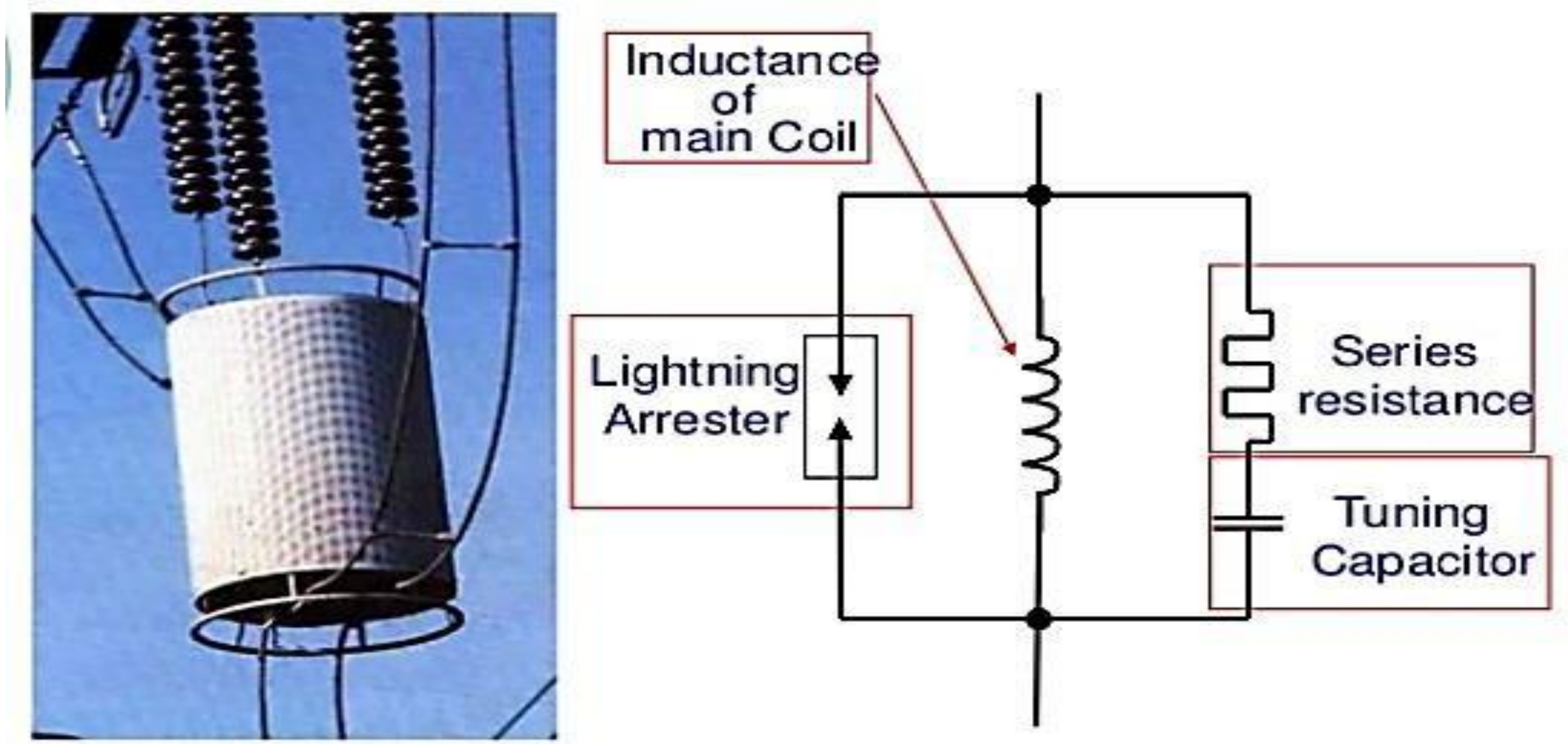


Electic Fan Circuit



Real Life Applications – Line Trap

Line Trap is a parallel LC circuit



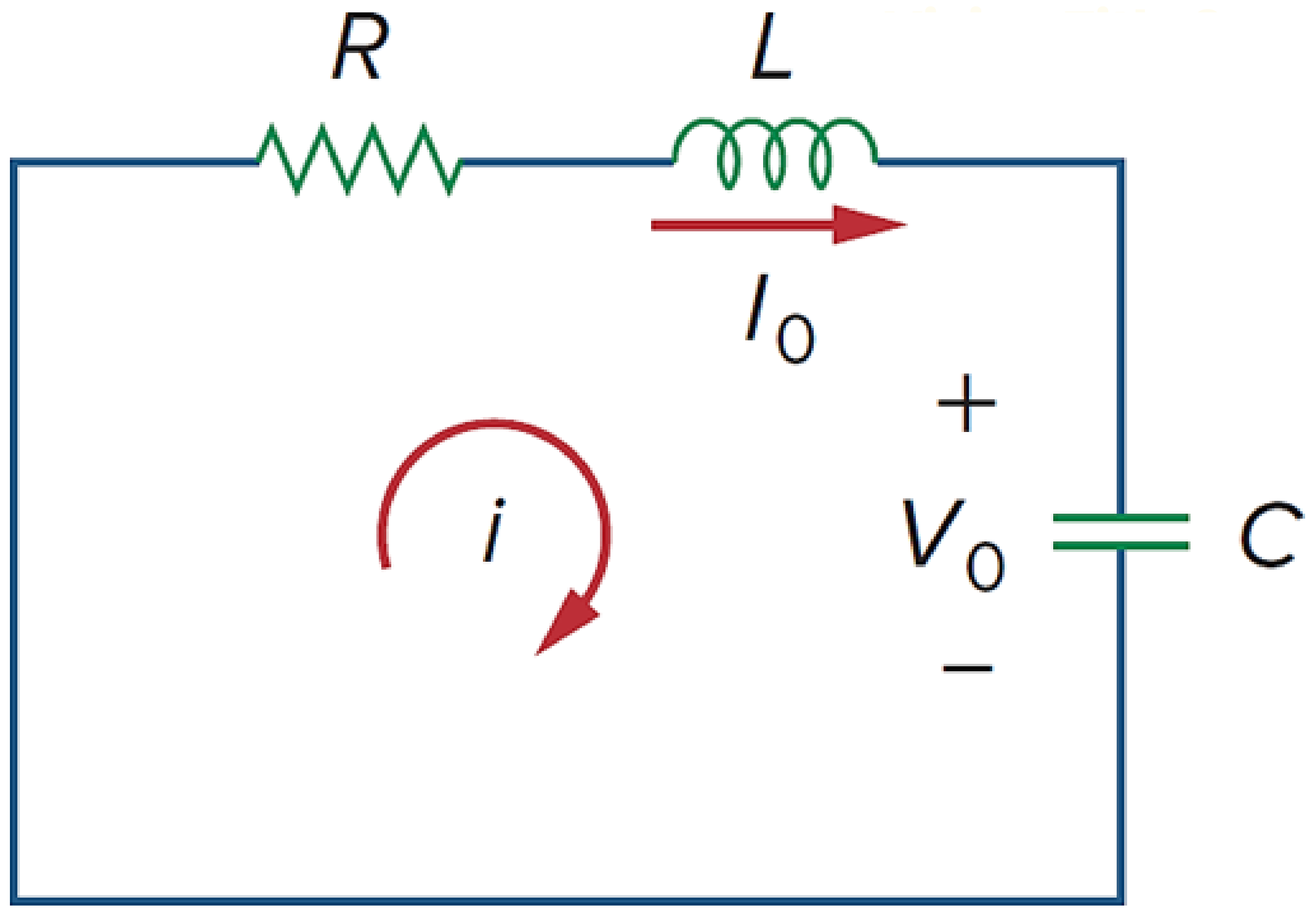
Step Response of an RLC Circuit

Consider the given series RLC circuit,

- The circuit is being excited by the energy initially stored in the capacitor and inductor.
- The energy is represented by the initial capacitor voltage V_0 and initial inductor current I_0.
- Thus, at $t = 0$,

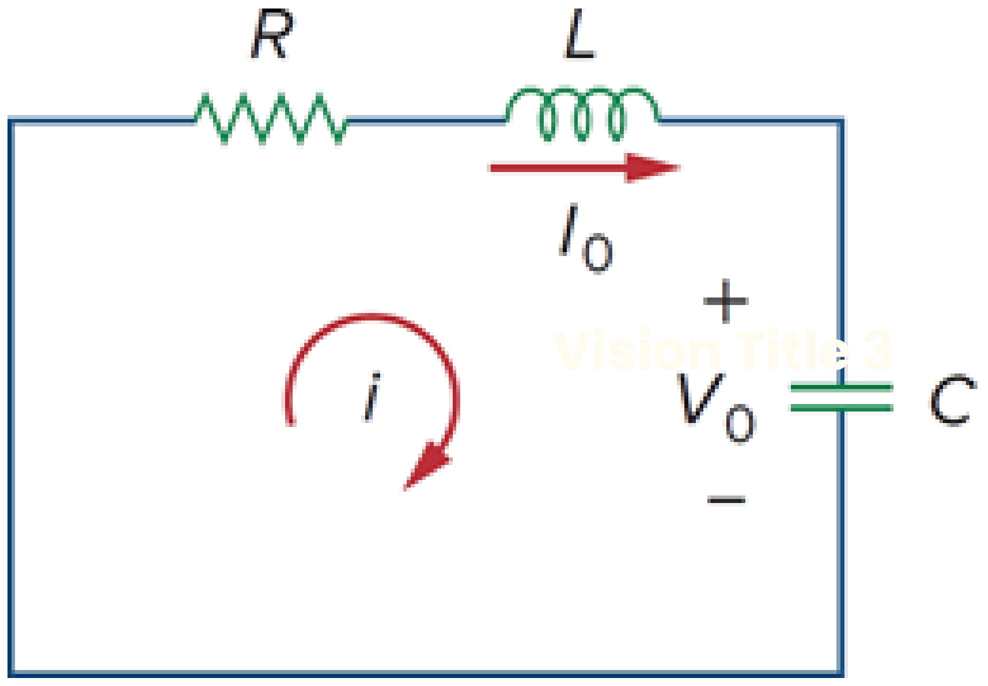
$$v(0) = \frac{1}{C}\int_{-\infty}^{0} i\, dt = V_0$$

$$i(0) = I_0$$



Step Response of an RLC Circuit

On applying KVL, $Ri + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{t} i(\tau)\, d\tau = 0$



on differentiating,

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$$

This is a second-order differential equation for the current i in the circuit.

The initial values and the first derivative are related as,

$$Ri(0) + L\frac{di(0)}{dt} + V_0 = 0; \quad \frac{di(0)}{dt} = -\frac{1}{L}(RI_0 + V_0)$$

Step Response of an RLC Circuit

Look for solutions of the form $i = Ae^{st}$ where, A and s are constants.

Substitute this into differential equation, $As^2e^{st} + \frac{AR}{L}se^{st} + \frac{Ase^{st}}{LC} = 0$

$$Ae^{st}\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right) = 0$$

Thus,

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

This quadratic equation is known as the characteristic equation

Step Response of an RLC Circuit

The roots of the equation dictate the character of i and they are given as,

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

A more compact way of expressing the roots is,

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

where,

$$\alpha = \frac{R}{2L}; \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

Step Response of an RLC Circuit

- The roots s_1 and s_2 are called **natural frequencies**, measured in nepers per second (Np/s), because they are associated with the natural response of the circuit
- ω_0 is known as the **resonant frequency** or strictly as the **undamped natural frequency**, expressed in radians per second (rad/s);
- α is the **neper frequency (or damping constant)** expressed in nepers per second.

The expression given is modified in terms of α and ω_0,

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

$$\boxed{s^2 + 2\alpha s + \omega_0^2 = 0}$$

Step Response of an RLC Circuit

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Step Response of an RLC Circuit

The two values of s indicate that there are two possible solutions for i,

$$i_1 = A_1 e^{s_1 t}; i_2 = A_2 e^{s_2 t}$$

- A complete or total solution would therefore require a linear combination of i_1 and i_2.
- Thus, the natural response of the series RLC circuit is

$$\boxed{i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}}$$

where the constants A_1 and A_2 are determined from the initial values $i(0)$ and $di(0)/dt$.

Three types of solutions are inferred:

1. If $\alpha > \omega_0$, we have the **over-damped** case.
2. If $\alpha = \omega_0$, we have the **critically-damped** case.
3. If $\alpha < \omega_0$, we have the **under-damped** case.

Step Response of an RLC Circuit

- **Overdamped Case ($\alpha > \omega_0$)**
  $\alpha > \omega_0$ implies $R^2 > 4L/C$

$$\alpha = \frac{R}{2L}; \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

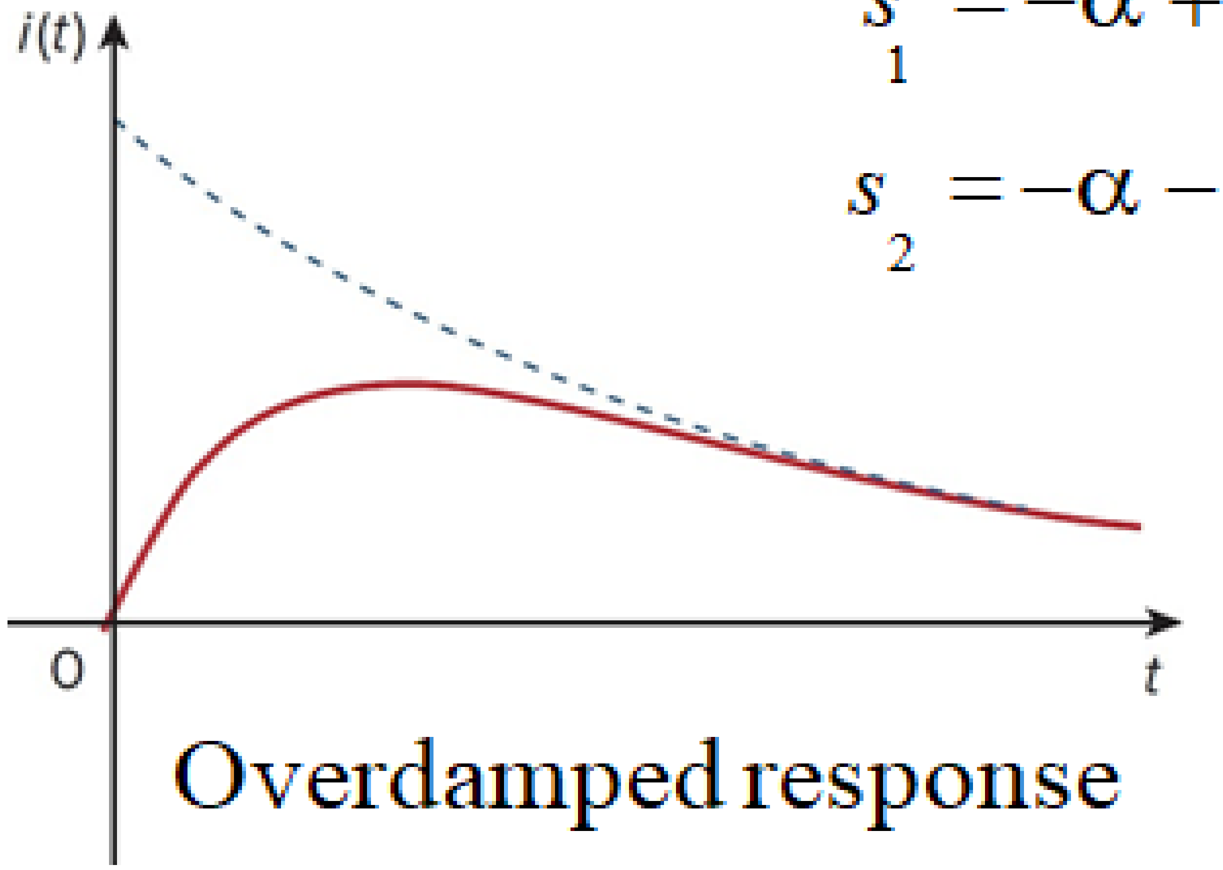
When this happens, both roots s_1 and s_2 are negative and real.

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

The response is given as,

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$



Overdamped response

Step Response of an RLC Circuit

- **Critically Damped Case ($\alpha = \omega_0$)**

$\alpha = \omega_0$ implies $R^2 = 4L/C$ Thus $s_1 = s_2 = -\alpha = -\dfrac{R}{2L}$

For this case,

$$i(t) = A_1 e^{-\alpha t} + A_2 e^{-\alpha t} = A_3 e^{-\alpha t}$$

where $A_3 = A_1 + A_2$. But this cannot be the solution, because the two initial conditions cannot be satisfied with the single constant A_3.

When $\alpha = \omega_0 = R/2L$, then

$$\frac{d^2 i}{dt^2} + 2\alpha \frac{di}{dt} + \alpha^2 i = 0$$

$$\frac{d}{dt}\left(\frac{di}{dt} + \alpha i\right) + \alpha\left(\frac{di}{dt} + \alpha i\right) = 0$$

let, $f = \dfrac{di}{dt} + \alpha i$, then,

$$\frac{df}{dt} + \alpha f = 0$$

this is a first-order differential equation with solution $f = A_1 e^{-\alpha t}$, where A_1 is a constant.

Step Response of an RLC Circuit

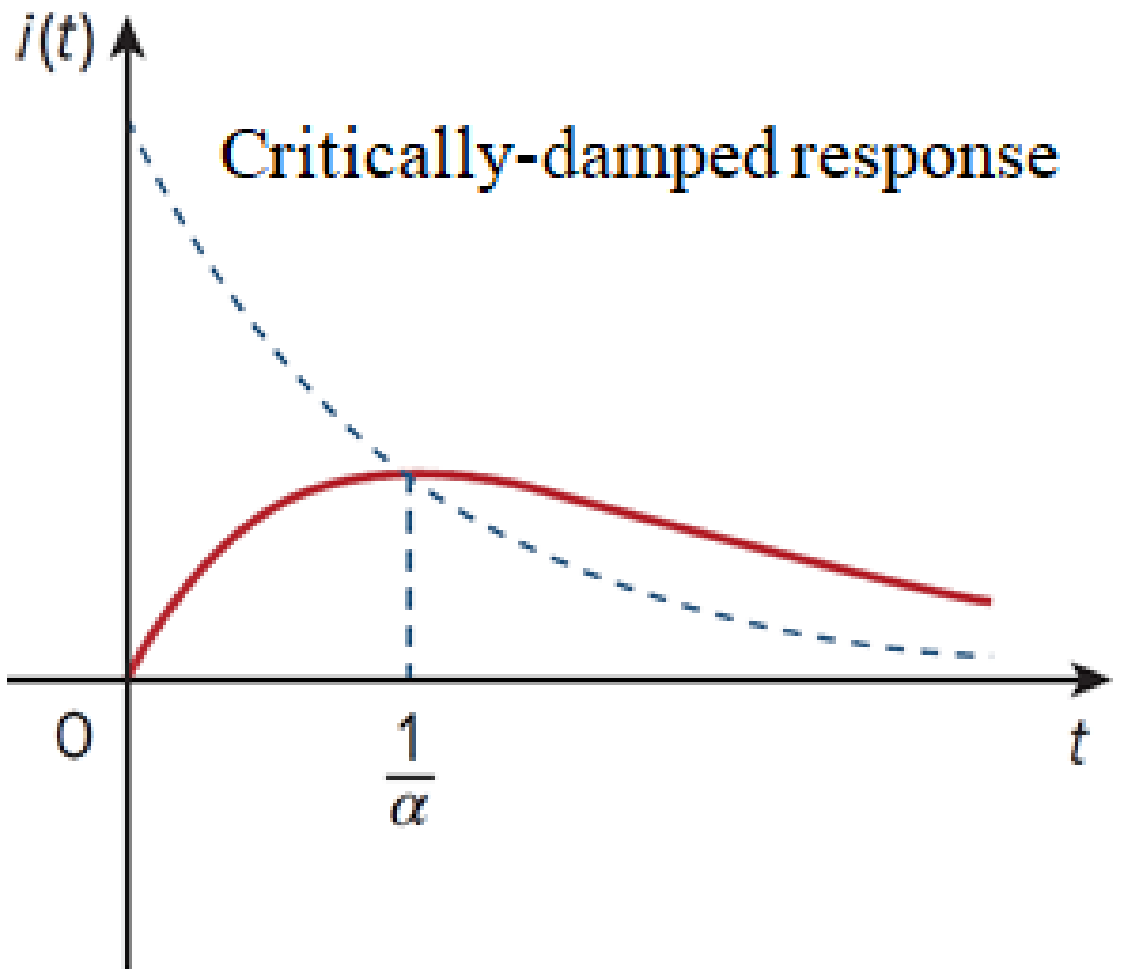
The original equation for current i becomes,

$$\frac{di}{dt} + \alpha i = A_1 e^{-\alpha t}$$

$$e^{\alpha t}\frac{di}{dt} + e^{\alpha t}\alpha i = A_1; \quad \frac{d}{dt}\left(e^{\alpha t} i\right) = A_1$$

on integration,

$$e^{\alpha t} i = A_1 t + A_2; \quad \boxed{i = (A_1 t + A_2)e^{-\alpha t}}$$



Hence, the natural response of the critically damped circuit is a sum of two terms: a negative exponential and a negative exponential multiplied by a linear term.

Step Response of an RLC Circuit

- **Underdamped Case ($\alpha < \omega_0$)**
  $\alpha < \omega_0$ implies $R^2 < 4L/C$. The roots may be written as,

$$s_1 = -\alpha + \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha + j\omega_d$$
$$s_2 = -\alpha - \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha - j\omega_d$$

where, $j = \sqrt{-1}$; $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$

Both ω_0 and ω_d are natural frequencies because they help determine the natural response; while ω_0 is called the ***undamped natural frequency***, ω_d is called the ***damped natural frequency***. The natural response is

$$i(t) = A_1 e^{-(\alpha - j\omega_d)t} + A_2 e^{-(\alpha + j\omega_d)t}$$
$$= e^{-\alpha t}\left(A_1 e^{-j\omega_d t} + A_2 e^{-j\omega_d t}\right)$$

Step Response of an RLC Circuit

By using Euler's identities,

$$i(t) = e^{-\alpha t}[A_1(\cos\omega_d t + j\sin\omega_d t) + A_2(\cos\omega_d t - j\sin\omega_d t)]$$

$$= e^{-\alpha t}[(A_1 + A_2)\cos\omega_d t + j(A_1 - A_2)\sin\omega_d t]$$

$$\boxed{i(t) = e^{-\alpha t}[B_1\cos\omega_d t + B_2\sin\omega_d t]}$$

Note:

It is clear that the natural response for this case is exponentially damped but also oscillatory in nature. The response has a time constant of $1/\alpha$ and a period of $T = 2\pi/\omega_d$.

Under-damped response