DEPARTMENT OF MATHEMATICS

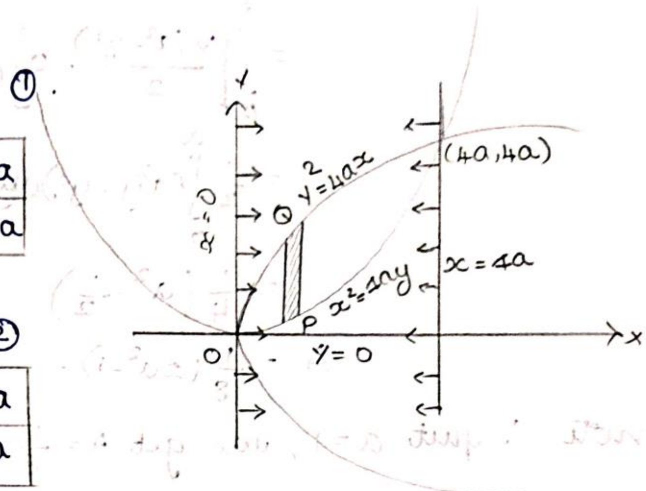
5. Show that the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is $\frac{16}{3}a^2$

Sol

Gn

$y^2 = 4ax$ — (1)

x	0	a	$4a$
$y = \pm 2\sqrt{ax}$	0	$\pm 2a$	$\pm 4a$



Gn $x^2 = 4ay$ — (2)

x	0	a	$4a$
$y = \frac{x^2}{4a}$	0	$\frac{a}{4}$	$4a$

Therefore, the point of intersection of (1) and (2) is (0, 0) and (4a, 4a).

Divide the area into vertical strip of width δx.

x varies from $x = 0$ to $x = 4a$ (vertical path).

y varies from $y = \frac{x^2}{4a}$ to $y = 2\sqrt{ax}$ (vertical strip)

$\therefore$ The required area $= \int_0^{4a} \int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy\,dx$

$$= \int_0^{4a} [y]_{y=\frac{x^2}{4a}}^{y=2\sqrt{ax}} dx$$

$$= \int_0^{4a} \left[2\sqrt{ax} - \frac{x^2}{4a}\right] dx$$

$$= \int_0^{4a} \left[2\sqrt{a}\cdot x^{1/2} - \frac{1}{4a} x^2\right] dx$$

$$= \left[2\sqrt{a}\,\frac{x^{3/2}}{\frac{3}{2}} - \frac{1}{4a}\frac{x^3}{3}\right]_0^{4a} = \left(\frac{4}{3}\sqrt{a}\,(4a)^{3/2} - \frac{(4a)^3}{(4a)^3}\right) - (0-0)$$

$$= \left[\frac{4}{3}\sqrt{a}\,(2\sqrt{a})^3 - \frac{(4a)^2}{3}\right]$$

$$= \frac{32}{3}a^2 - \frac{16a^2}{3} = \frac{16}{3}a^2 \text{ square units}$$

Area enclosed by plane curves

1. Find the area of the lemniscate $r^2 = a^2\cos 2\theta$ by double integration.

Sol.

Area = 4 × area of upper half of one loop.

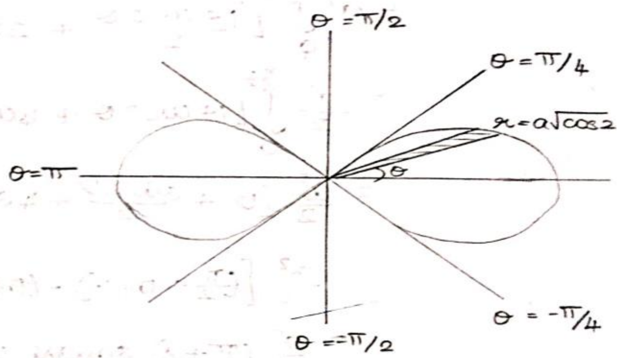
$$= 4\int_0^{\pi/4}\int_0^{a\sqrt{\cos 2\theta}} r\,dr\,d\theta$$

$$= 2\int_0^{\pi/4} (r^2)_0^{a\sqrt{\cos 2\theta}}$$

$$= 2a^2\int_0^{\pi/4} \cos 2\theta\, d\theta$$

$$= 2a^2\left(\frac{\sin 2\theta}{2}\right)_0^{\pi/4}$$

$$= a^2 \text{ square units}$$



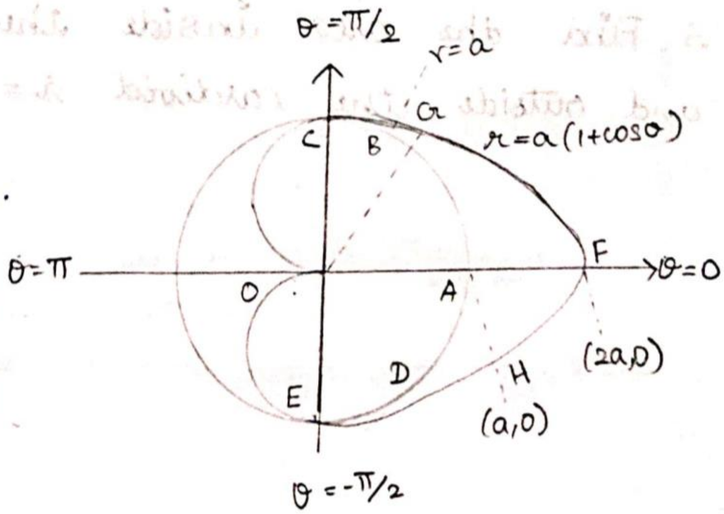
2. Find the area that lies inside the cardioid $r = a(1+\cos\theta)$ and outside the circle $r = a$, by double integration.

Sol

Gn

$r = a(1+\cos\theta)$, $r = a$.

θ	0°	$\frac{\pi}{2}$
$r = a(1+\cos\theta)$	$r = 2a$	$r = a$
$r = a$	$r = a$	$r = a$



Both the curves are symmetric about the initial line. Hence, the required area.

$= 2 \times AFGCBA.$

$$= 2\int_0^{\pi/2}\int_a^{a(1+\cos\theta)} r\,dr\,d\theta = 2\int_0^{\frac{\pi}{2}}\left[\frac{r^2}{2}\right]_a^{a(1+\cos\theta)} d\theta.$$

$$= 2\int_0^{\pi/2}\left[\frac{a^2(1+\cos\theta)^2}{2} - \frac{a^2}{2}\right] d\theta.$$

$$= a^2\int_0^{\frac{\pi}{2}}[1+\cos^2\theta + 2\cos\theta - 1]\,d\theta.$$

$$= a^2\int_0^{\pi/2}\left[\frac{1+\cos 2\theta}{2} + 2\cos\theta\right] d\theta.$$

$$= \frac{a^2}{2}\int_0^{\pi/2}[1+\cos 2\theta + 4\cos\theta]\,d\theta$$

$$= \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2} + 4\sin\theta\right]_0^{\pi/2}$$

$$= \frac{a^2}{2}\left[\left(\frac{\pi}{2}+0+4\right) - (0+0+0)\right]$$

$$= \frac{a^2}{4}(\pi+8) \text{ square units}$$

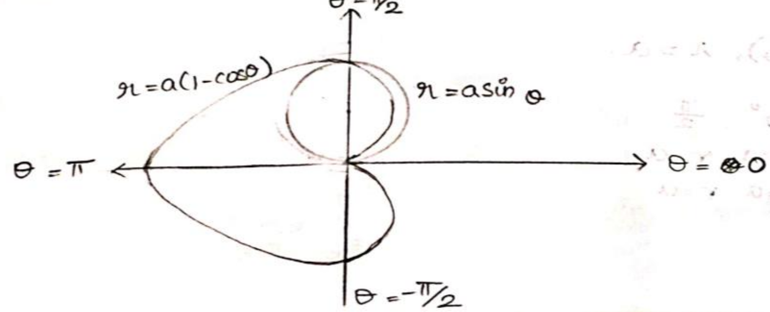
When $r = 2$

put $a = 2$,

we get required area $= \pi + 8$ square unit.

---

3. Find the area inside the circle $r = a\sin\theta$ and outside the cardioid $r = a(1-\cos\theta)$.



θ	0°	$\frac{\pi}{2}$	π
$r = a\sin\theta$	0	a	0
$r = a(1-\cos\theta)$	0	a	$2a$

From the figures, we get.

θ varies from $\theta = 0$ to $\theta = \frac{\pi}{2}$.

r varies from $r = a(1-\cos\theta)$ to $r = a\sin\theta$

$$\therefore \text{The required area} = \int_0^{\pi/2}\int_{a(1-\cos\theta)}^{a\sin\theta} r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left[\frac{r^2}{2}\right]_{a(1-\cos\theta)}^{a\sin\theta} d\theta$$

$$= \int_0^{\pi/2}\left[\frac{a^2\sin^2\theta}{2} - \frac{a^2(1-\cos\theta)^2}{2}\right] d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2}[\sin^2\theta - (1+\cos^2\theta - 2\cos\theta)]\,d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2}[\sin^2\theta - 1 - \cos^2\theta + 2\cos\theta]\,d\theta$$

$$= \frac{a^2}{2}\left[\int_0^{\frac{\pi}{2}}\sin^2\theta\,d\theta - \int_0^{\pi/2}d\theta - \int_0^{\pi/2}\cos^2\theta\,d\theta + 2\int_0^{\pi/2}\cos\theta\,d\theta\right]$$

$$= \frac{a^2}{2}\left[-[\theta]_0^{\pi/2} + 2[\sin\theta]_0^{\pi/2}\right]$$

$$\boxed{\because \int_0^{\pi/2}\sin^2\theta\,d\theta = \int_0^{\pi/2}\cos^2\theta\,d\theta = \frac{1}{2}\cdot\frac{\pi}{2}}$$

$$= \frac{a^2}{2}\left[-\left(\frac{\pi}{2}-0\right) + 2(1-0)\right]$$

$$= \frac{a^2}{2}\left[2 - \frac{\pi}{2}\right]$$

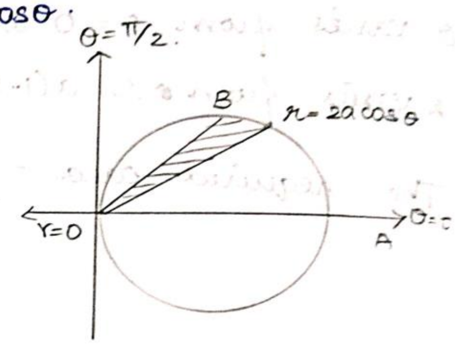
$$= \frac{a^2}{4}(4-\pi) \text{ square units}$$

4. Evaluate : $\iint r^2 \sin\theta \, dr\, d\theta$ where R is the semi circle $r = 2a\cos\theta$ about the initial line.

Sol :

Here, the region is defined by.

$0 \le \theta \le \frac{\pi}{2}, \quad 0 \le r \le 2a\cos\theta$



Let $I = \iint r^2 \sin\theta \, dr\, d\theta$

$$= \int_0^{\pi/2} \int_0^{2a\cos\theta} (r^2 \sin\theta)\, dr\, d\theta.$$

$$= \int_0^{\pi/2} \left[\frac{r^3}{3}\sin\theta\right]_{r=0}^{r=2a\cos\theta} d\theta$$

$$= \int_0^{\pi/2} \left[\frac{8a^3}{3}\cos^3\theta \sin\theta - 0\right] d\theta$$

$$= \int_0^{\pi/2} \frac{8a^3}{3}\cos^3\theta \sin\theta \, d\theta$$

$$= \frac{8a^3}{3}\int_0^{\pi/2} \cos^3\theta \sin\theta \, d\theta.$$

$$= \frac{-8a^3}{3}\int_0^{\pi/2} \cos^3\theta \, d[\cos\theta]$$

$$= \frac{-8a^3}{3}\left[\frac{\cos^4\theta}{4}\right]_0^{\pi/2}$$

$$= \frac{-8a^3}{3}\left[0 - \frac{1}{4}\right] = \frac{2a^3}{3}$$