Area enclosed by plane curves:

1. Evaluate $\iint_R xy\,dx\,dy$, where R is the domain bounded by x axis, ordinate $x = 2a$ and the curve $x^2 = 4ay$.

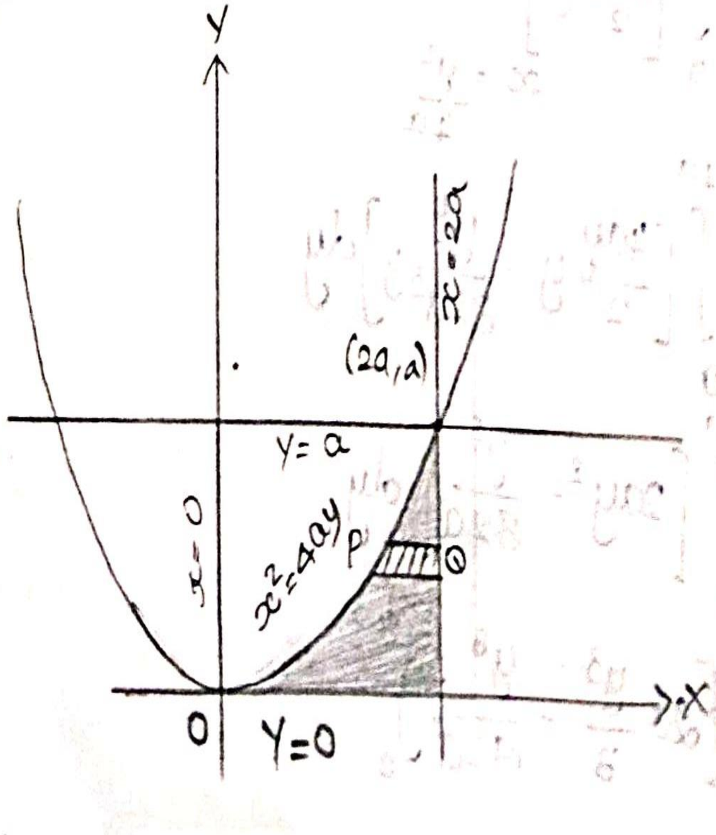
Sol

Given x-axis => $y = 0$ line

$x = 2a$

$x^2 = 4ay$

$y = \dfrac{x^2}{4a}$.



x	0	$2a$
y	0	a

Here, y limits varies from $y=0$ to $y=a$ (Horizontal path).

x limits varies from $x = 2\sqrt{ay}$ to $x = 2a$ (Horizontal strip PQ).

$$\therefore \text{The required area} = \int_0^a \int_{2\sqrt{ay}}^{2a} xy \, dx \, dy.$$

$$= \int_0^a \left[y\frac{x^2}{2} \right]_{x=2\sqrt{ay}}^{x=2a} dy$$

$$= \int_0^a \left[\frac{4a^2 y}{2} - \frac{4ay^2}{2} \right] dy = \int_0^a [2a^2 y - 2ay^2] \, dy.$$

$$= \left[2a^2 \frac{y^2}{2} - 2a\frac{y^3}{3} \right]_0^a = \left(a^4 - \frac{2a^4}{3} \right) - (0-0)$$

$$= \frac{1}{3}a^4 \text{ square units}$$

2. Using double integral, find the area bounded by $y=x$ and $y=x^2$

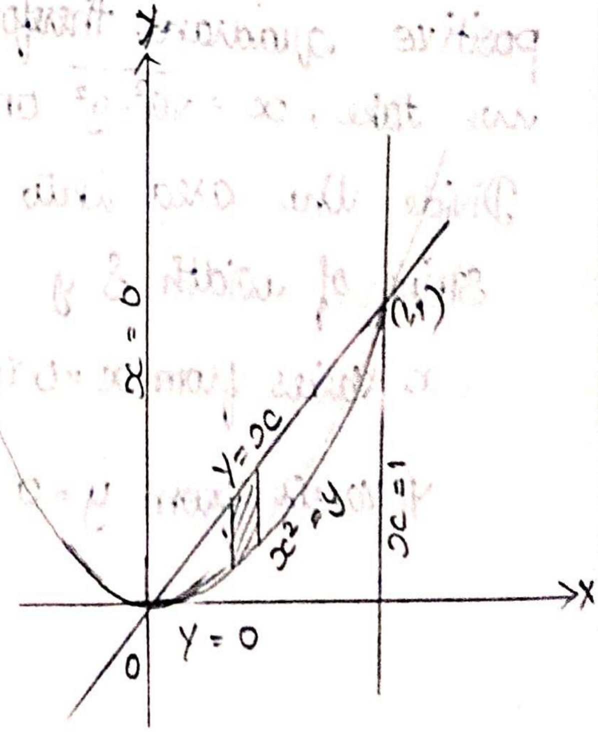
Sol

Gn: $y = x$ — (1)

x	0	1	2	-1	-2
$y=x$	0	1	2	-1	-2

Gn. $y = x^2$ — (2)

x	0	1	2	-1	-2
$y=x^2$	0	1	4	1	4



Therefore, the point of intersection of (1) and (2) is (0,0) and (1,1).

Divide the area into vertical strip of width δx.

x varies from $x = 0$ to $x = 1$ (vertical path)

y varies from $y = x^2$ to $y = x$ (vertical strip PQ)

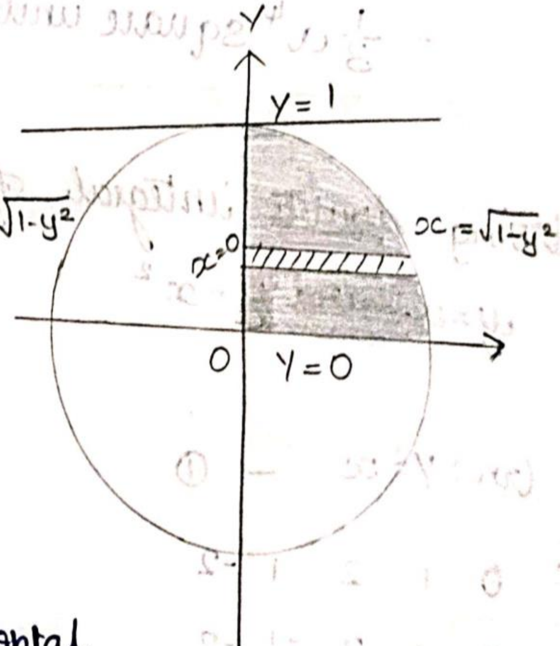
$$\text{The required area} = \int_0^1 \int_{x^2}^{x} dy\,dx$$

$$= \int_0^1 [y]_{y=x^2}^{y=x} dx = \int_0^1 (x - x^2)\,dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \left(\frac{1}{2} - \frac{1}{3}\right) - (0-0)$$

$$= \frac{3-2}{6} = \frac{1}{6} \text{ square unit.}$$

3. Evaluate : $\iint xy\, dx\, dy$ over the positive quadrant of the circle $x^2+y^2=a^2$.



Sol.

Gn:

$$x^2+y^2=a^2$$

$$x^2=a^2-y^2$$

$$x = \pm\sqrt{a^2+y^2}$$

positive quadrant therefore.
we take, $x=\sqrt{a^2-y^2}$ only.

Divide the area into horizontal strips of width δy

x varies from $x=0$ to $x=\sqrt{a^2-y^2}$

y varies from $y=0$ to $y=a$

$\therefore$ The required area $= \int_0^a \int_0^{\sqrt{a^2-y^2}} xy\,dx\,dy$

$$= \int_0^a \left[y\,\frac{x^2}{2}\right]_{x=0}^{x=\sqrt{a^2-y^2}} dy.$$

$$= \int_0^a \left[\frac{y(a^2-y^2)}{2} - 0\right] dy$$

$$= \frac{1}{2}\int_0^a (a^2y - y^3)\,dy = \frac{1}{2}\left[\frac{a^2y^2}{2} - \frac{y^4}{4}\right]_0^a$$

$$= \frac{1}{4}\left(a^2 - \frac{1}{2}\right)$$

$$A = \frac{1}{8}(2a^2-1)$$

note1: put $a=1$, we get $A = \frac{1}{8}$, put $a=2$, we get $A = \frac{7}{8}$

H.W

4. Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

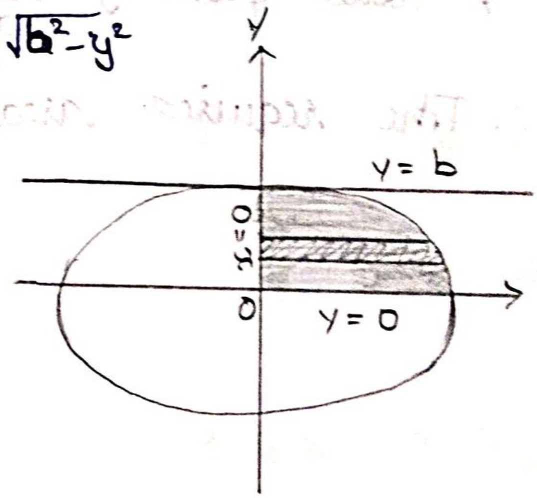
Sol

Area of ellipse = 4 x area of quadrant.

Divide the area into horizontal strips of width δy

x varies from $x=0$ to $x=\frac{a}{b}\sqrt{b^2-y^2}$

y varies from $y=0$ to $y=b$



$\therefore$ The required area

$$= 4\int_0^b \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} dx\,dy$$

$$= 4\int_0^b [x]_0^{\frac{a}{b}\sqrt{b^2-y^2}}\,dy$$

$$= 4\int_0^b \left[\frac{a}{b}\sqrt{b^2-y^2} - 0\right] dy.$$

$$= \frac{4a}{b}\int_0^b \sqrt{b^2-y^2}\,dy = \frac{4a}{b}\left[\frac{b^2}{2}\sin^{-1}\frac{y}{b} + \frac{y}{2}\sqrt{b^2-y^2}\right]_0^b$$

$$= \frac{4a}{b}\left[\left[\frac{b^2}{2}\frac{\pi}{2}+0\right]-(0+0)\right] = \frac{4a}{b}\,\frac{b^2}{2}\cdot\frac{\pi}{2}$$

$$= \pi ab \text{ square units}$$