## Volume of Solids

1. Find the volume of the sphere $x^2+y^2+z^2=a^2$ without transformation.

**Sol**

$V = 8 \times$ Volume in an octant

$z$ varies from $z=0$ to $z=\sqrt{a^2-x^2-y^2}$

$y$ varies from $y=0$ to $y=\sqrt{a^2-x^2}$

$x$ varies from $x=0$ to $x=a$.

$$V = 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx.$$

$$= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\sqrt{a^2-x^2-y^2} - 0\right] dy\,dx.$$

$$= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\; dy\,dx.$$

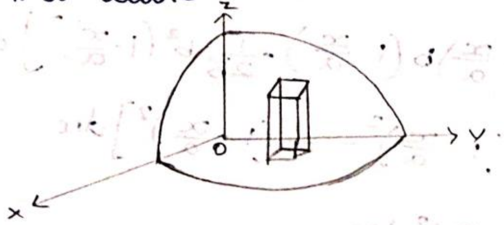
$$= 8\int_0^a \left[\frac{a^2-x^2}{2}\sin^{-1}\frac{y}{\sqrt{a^2-x^2}} + \frac{y}{2}\sqrt{a^2-x^2-y^2}\right]_{y=0}^{y=\sqrt{a^2-x^2}} dx$$

$$= 8\int_0^a \left[\frac{a^2-x^2}{2}\,\frac{\pi}{2}\right] dx$$

$$= 2\pi\int_0^a (a^2-x^2)\,dx \quad = 2\pi\left[a^2x - \frac{x^3}{3}\right]_0^a$$

$$= 2\pi\left[\left(a^3-\frac{a^3}{3}\right) - (0-0)\right] = 2\pi\left[\frac{2}{3}a^3\right]$$

$$= \frac{4}{3}\pi a^3 \text{ cubic units.}$$



2. Find the volume of the tetrahedron bounded by the planes $x=0, y=0, z=0$ and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

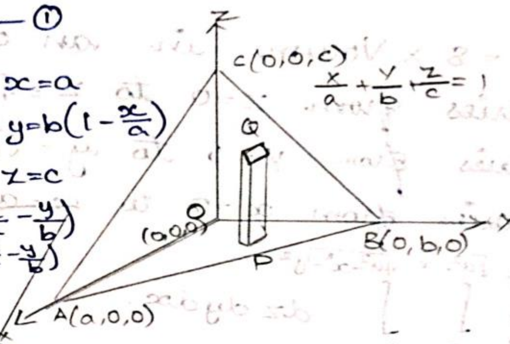
Sol.

let $V = \iiint_V dz\,dy\,dx$ — ①

limits of $x$ are $x=0$ to $x=a$

limits of $y$ are $y=0$ to $y=b\left(1-\frac{x}{a}\right)$

limits of $z$ are $z=0$ to $z=c\left(1-\frac{x}{a}-\frac{y}{b}\right)$




$$(1) \Rightarrow V = \int_0^a \int_0^{b(1-\frac{x}{a})} [z]_0^{c(1-\frac{x}{a}-\frac{y}{b})}$$

$$\cancel{= \int_0^a \int_0^{b(1-\frac{x}{a})} c\left(1-\frac{x}{a}-\frac{y}{b}\right) dy\,dx}$$

$$= \int_0^a \int_0^{b(1-\frac{x}{a})} [z]$$

$$① \Rightarrow V = \int_0^a \int_0^{b(1-\frac{x}{a})} \int_0^{c(1-\frac{x}{a}-\frac{y}{b})} dz\,dy\,dx.$$

$$= \int_0^a \int_0^{b(1-\frac{x}{a})} [z]_0^{c(1-\frac{x}{a}-\frac{y}{b})} dy\,dx.$$

$$= \int_0^a \int_0^{b(1-\frac{x}{a})} c\left(1-\frac{x}{a}-\frac{y}{b}\right) dy\,dx$$

$$= c\int_0^a \left[\left(1-\frac{x}{a}\right)y - \frac{1}{b}\frac{y^2}{2}\right]_0^{b(1-\frac{x}{a})} dx$$

$$= c\int_0^a \left[\left(1-\frac{x}{a}\right)b\left(1-\frac{x}{a}\right) - \frac{1}{2\cancel{b}}b^2\left(1-\frac{x}{a}\right)^2\right] dx$$

$$= bc\int_0^a \left[\left(1-\frac{x}{a}\right)^2 - \frac{1}{2}\left(1-\frac{x}{a}\right)^2\right] dx$$

$$= \frac{bc}{2}\int_0^a \left(1-\frac{x}{a}\right)^2 dx$$

$$= \frac{bc}{2}\left[\frac{\left(1-\frac{x}{a}\right)^3}{3\cdot\left(\frac{-1}{a}\right)}\right]_0^a = -\frac{abc}{6}\left[\left(1-\frac{x}{a}\right)^3\right]_0^a$$

$$= \frac{-abc}{6}(0-1) = \frac{abc}{6} \text{ cubic units.}$$

Note :

① Evaluate $\iiint_V dx\,dy\,dz$, where $V$ is volume of the tetrahedron whose vertices are $(0,0,0)$ $(0,1,0)$, $(1,0,0)$ and $(0,0,1)$.

put $a=1$, $b=1$, $c=1$, then we get $V = \frac{1}{6}$ cubic unit

② Evaluate $\iiint_V dx\,dy\,dz$, where $V$ is the volume of the tetrahedron whose vertices are $(0,0,0)$ $(a,0,0)$, $(0,a,0)$ and $(0,0,a)$.

Put $b=a$ and $c=a$, then we get $V = \frac{a^3}{6}$ cubic units.

③ Find the volume of the tetrahedron in space cut from the first octant by the plane $6x+3y+2z=6$

Hint : Gn

$$6x+3y+2z=6. \quad \div 6 \Rightarrow \frac{x}{1}+\frac{y}{2}+\frac{z}{3}=1$$

put $a=1$, $b=2$, $c=3$ then we get $V = \frac{(1)(2)(3)}{6} = 1$ cubic unit

④ Evaluate $\iiint_V dx\,dy\,dz$, where $V$ is the finite region of space (tetrahedron) bounded by the planes $x=0$, $y=0$, $z=0$ and $2x+3y+4z=12$.

Hint : Given :

$$2x+3y+4z=12 \quad \div 12 \Rightarrow \frac{x}{6}+\frac{y}{4}+\frac{z}{3}=1$$

put :

$a=6$, $b=4$, $c=3$ then, we get $V = \frac{(6)(4)(3)}{6} = 12$ cubic units