4. FUNCTIONS OF SEVERAL VARIABLE

PARTIAL DIFFERENTIATION

Let $U = f(x, y)$ be a function of two independent variables.

Differentiate U w.r.t 'x', considering 'y' as constant ~~This~~ is known as partial differential coefficient of U w.r.t x. It is denoted by $\frac{\partial u}{\partial x}$.

Similarly, if we differtiate U w.r.t 'y', considering 'x' as constant is known as partial differential coefficient of U w.r.t y. It is denoted as $\frac{\partial U}{\partial y}$.

1. Find the 1st and 2nd derivative of

$U = x^3 + y^3 - 3axy$

Ist order.

$$\frac{\partial U}{\partial x} = 3x^2 - 3ay$$

$$\frac{\partial u}{\partial y} = 3y^2 - 3ax$$

II nd Order.

$$\frac{\partial^2 u}{\partial x^2} = 6x \qquad \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = -3a$$

$$\frac{\partial^2 u}{\partial y^2} = 6y \qquad \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = -3a$$

2. Find the Ist order derivative of $u = xe^y + ye^x$

$$\frac{\partial u}{\partial x} = (1)e^y + ye^x(1) = e^y + ye^x$$

$$\frac{\partial u}{\partial y} = xe^y(1) + (1)e^x = xe^y + e^x$$

3. If $(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$, show that

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2} = \frac{2}{r}$$

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

Diff w.r.t x

$$\cancel{2}(x-a) = \cancel{2}r\frac{\partial r}{\partial x}$$

$$\frac{\partial r}{\partial x} = \frac{x-a}{r}$$

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$$\frac{\partial^2 r}{\partial x^2} = \frac{r(1) - (x-a)\frac{\partial r}{\partial x}}{r^2}$$

$$= \frac{r - (x-a)\left(\frac{x-a}{r}\right)}{r^2}$$

$$= \frac{r}{r^2} - \frac{(x-a)^2}{r^3}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{1}{r} - \frac{(x-a)^2}{r^3} \quad -(1)$$

Diff w.r.t y.

$$2(y-b) = 2r \cdot \frac{\partial r}{\partial y}$$

$$\frac{\partial r}{\partial y} = \frac{y-b}{r}$$

$$\frac{\partial^2 r}{\partial y^2} = \frac{r(1) - (y-b)\left(\frac{\partial r}{\partial y}\right)}{r^2}$$

$$= \frac{r - (y-b)\left(\frac{y-b}{r}\right)}{r^2}$$

$$= \frac{r}{r^2} - \frac{(y-b)^2}{r^3}$$

$$\frac{\partial^2 r}{\partial y^2} = \frac{1}{r} - \frac{(y-b)^2}{r^3} \quad -(2)$$

Diff. w.r.t z

$$2(z-c) = 2r \cdot \frac{\partial r}{\partial z}$$

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$$\frac{\partial^2 r}{\partial z^2} = \frac{r - (z-c)\left(\frac{z-c}{r}\right)}{r^2}$$

$$= \frac{r}{r^2} - \frac{(z-c)^2}{r^3}$$

$$\frac{\partial^2 r}{\partial z^2} = \frac{1}{r} - \frac{(z-c)^2}{r^3} \quad -(3)$$

(1) + (2) + (3)

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2} = \frac{1}{r} - \frac{(x-a)^2}{r^3} + \frac{1}{r} - \frac{(y-b)^2}{r^3} + \frac{1}{r} - \frac{(z-c)^2}{r^3}$$

$$= \frac{3}{r} - \frac{1}{r^3}\left[(x-a)^2 + (y-b)^2 + (z-c)^2\right]$$

$$= \frac{3}{r} - \frac{1}{r^3}(r^2)$$

$$= \frac{3}{r} - \frac{1}{r}$$

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2} = \frac{2}{r}$$

Hence proved.

4. If $z = f(x+ct) + g(x-ct)$ then prove

$$\frac{\partial^2 z}{\partial t^2} = c^2\frac{\partial^2 z}{\partial x^2}$$

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$z = f(x+ct) + g(x-ct)$ —(1)

Diff (1) w.r.t t.

$$\frac{\partial z}{\partial t} = f'(x+ct)(c) + g'(x-ct)(-c)$$

Again diff.

$$\frac{\partial^2 z}{\partial t^2} = f''(x+ct)(c)(c) + g''(x-ct)(-c)(-c)$$

$$= c^2\left[f''(x+ct) + g''(x-ct)\right] \quad -(2)$$

Diff (1) w.r.t x

$$\frac{\partial z}{\partial x} = f'(x+ct)(1) + g'(x-ct)(1)$$

Again diff.

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ct) + g''(x-ct) \quad -(3)$$

From (2) & (3)

$$\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$$

5. Verify that $u_{xy} = u_{yx}$

$$u = \tan^{-1}\left(\frac{x}{y}\right)$$

diff. $\tan^{-1} x$

$= \frac{1}{1+x^2}$

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Diff. w.r.t x

$$U_x = \frac{1}{1+\left(\frac{x}{y}\right)^2}\,\frac{1}{y}$$

$$= \frac{1}{\frac{y^2+x^2}{y^2}}\,\frac{1}{y}$$

$$= \frac{y^2}{x^2+y^2}\,\frac{1}{y}$$

$$U_x = \frac{y}{x^2+y^2} \quad -(1).$$

Diff. w.r.t y

$$U_y = \frac{1}{1+\left(\frac{x}{y}\right)^2}\left(\frac{-x}{y^2}\right)$$

$$= \frac{1}{\frac{x^2+y^2}{y^2}}\left(\frac{-x}{y^2}\right)$$

$$= \frac{y^2}{x^2+y^2}\left(\frac{-x}{y^2}\right)$$

$$U_y = \frac{-x}{x^2+y^2} \quad -(2)$$

Diff (2) w.r.t x.

$$U_{xy} = -\left[\frac{(x^2+y^2)\cdot(1) - x(2x)}{(x^2+y^2)^2}\right]$$

$$= -\left[\frac{x^2+y^2-2x^2}{(x^2+y^2)^2}\right]$$

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$$= -\left[\frac{y^2-x^2}{(x^2+y^2)^2}\right]$$

$$U_{xy} = \frac{x^2-y^2}{(x^2+y^2)^2} \quad -(3)$$

Diff (1) w.r.t y

$$U_{yx} = \left[\frac{(x^2+y^2)(1)-y(2y)}{(x^2+y^2)^2}\right]$$

$$= \left[\frac{x^2+y^2-2y^2}{(x^2+y^2)^2}\right]$$

$$U_{yx} = \frac{x^2-y^2}{(x^2+y^2)^2} \quad -(4)$$

From (3) & (4)

$$U_{xy} = U_{yx}$$

Hence proved.

6. If $U = e^{xy}$, prove $U_{xx} + U_{yy} = \frac{1}{U}\left[U_x^2 + U_y^2\right]$

$$U_x = e^{xy}(y)$$

$$U_x = ye^{xy}$$

$$U_{xx} = ye^{xy}(y)$$

$$U_{xx} = y^2 e^{xy} \quad -(1)$$

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$$U_y = x e^{xy}$$

$$U_{yy} = x e^{xy}(x)$$

$$U_{yy} = x^2 e^{xy} \quad -(2)$$

(1) + (2).

$$U_{xx} + U_{yy} = y^2 e^{xy} + x^2 e^{xy}$$

$$= U y^2 + U x^2 \quad -(3).$$

$$\frac{1}{U}\left[U_x^2 + U_y^2\right] = \frac{1}{e^{xy}}\left[e^{2xy} y^2 + x^2 e^{2xy}\right]$$

$$= \frac{e^{2xy}}{e^{xy}}\left[y^2 + x^2\right]$$

$$= e^{xy}\left[y^2 + x^2\right]$$

$$= U(x^2 + y^2)$$

$$\frac{1}{U}\left[U_x^2 + U_y^2\right] = Ux^2 + Uy^2 \quad -(4).$$

From (3) & (4)

$$U_{xx} + U_{yy} = \frac{1}{U}\left[U_x^2 + U_y^2\right]$$

Hence proved.

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