JACOBIAN.

If $u = f(x,y)$, $v = g(x,y)$ are the two functions in two variables, then the jacobian of u and v w.r.t x and y is denoted by J.

$$J = \frac{\partial(u,v)\ (\text{function})}{\partial(x,y)\ (\text{variable})} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Properties.

i) $JJ' = 1$

ii) If u, v and w are functionally dependent, then $J = 0$

1. If $u = e^x \cos y$, $v = e^x \sin y$, find J

$$J = \frac{\partial(u,v)}{\partial(x,y)}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$\frac{\partial u}{\partial x} = e^x \cos y$ $\quad$ $\frac{\partial u}{\partial y} = -e^x \sin y$

$\frac{\partial v}{\partial x} = e^x \sin y$ $\quad$ $\frac{\partial v}{\partial y} = e^x \cos y$

$$J = \begin{vmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{vmatrix}$$

$$= e^{2x}(\cos^2 y) + e^{2x} \sin^2 y$$

$$= e^{2x}(\cos^2 y + \sin^2 y)$$

$$J = e^{2x}$$

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2. $x = \frac{u^2}{v}$ $\quad y = \frac{v^2}{w}$ $\quad z = \frac{w^2}{u}$

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)}$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$\frac{\partial x}{\partial u} = \frac{2u}{v} \qquad \frac{\partial x}{\partial v} = \frac{-u^2}{v^2} \qquad \frac{\partial x}{\partial w} = 0$$

$$\frac{\partial y}{\partial u} = 0 \qquad \frac{\partial y}{\partial v} = \frac{2v}{w} \qquad \frac{\partial y}{\partial w} = \frac{-v^2}{w^2}$$

$$\frac{\partial z}{\partial u} = \frac{-w^2}{u^2} \qquad \frac{\partial z}{\partial v} = 0 \qquad \frac{\partial z}{\partial w} = \frac{2w}{u}$$

$$J = \begin{vmatrix} \frac{2u}{v} & -\frac{u^2}{v^2} & 0 \\ 0 & \frac{2v}{w} & \frac{-v^2}{w^2} \\ \frac{-w^2}{u^2} & 0 & \frac{2w}{u} \end{vmatrix}$$

$$= \frac{2u}{v}\left[\frac{4vw}{u\cancel{w}} + 0\right] + \frac{u^2}{v^2}\left[0 - \frac{v^2w^2}{u^2w^2}\right] + 0$$

$$= \frac{8uv}{uv} - \frac{u^2v^2}{u^2v^2}$$

$$= 8 - 1$$

$$= 7$$

3. If $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$, find Jacobian

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$\frac{\partial u}{\partial x} = \frac{-yz}{x^2}$ $\quad \frac{\partial u}{\partial y} = \frac{z}{x}$ $\quad \frac{\partial u}{\partial z} = \frac{y}{x}$

$\frac{\partial v}{\partial x} = \frac{z}{y}$ $\quad \frac{\partial v}{\partial y} = \frac{-zx}{y^2}$ $\quad \frac{\partial v}{\partial z} = \frac{x}{y}$

$\frac{\partial w}{\partial x} = \frac{y}{z}$ $\quad \frac{\partial w}{\partial y} = \frac{x}{z}$ $\quad \frac{\partial w}{\partial z} = \frac{-xy}{z^2}$

$$J = \begin{vmatrix} \frac{-yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & \frac{-zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & \frac{-xy}{z^2} \end{vmatrix}$$

$$= \frac{1}{x^2y^2z^2} \begin{vmatrix} -yz & zx & yx \\ zy & -zx & xy \\ yz & xz & -xy \end{vmatrix}$$

$$= \frac{yz \cdot xz \cdot xy}{x^2y^2z^2} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

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$$= \frac{x^2 \cdot y^2 \cdot z^2}{x^2 \cdot y^2 \cdot z^2}\left[-1(0) - 1(-2) + 1(2)\right]$$

$$= +2 + 2$$

$$J = 4$$

4. Show that $u = \frac{x}{y}$, $v = \frac{x+y}{x-y}$ are functionally dependent. Find the relation between them. ↳ $J = 0$

$$J = \frac{\partial(u, v)}{\partial(x, y)}$$

$$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$\frac{\partial u}{\partial x} = \frac{1}{y} \qquad \frac{\partial u}{\partial y} = \frac{-x}{y^2}$$

$$\frac{\partial v}{\partial x} = \frac{(x-y)(1) - (x+y)(1)}{(x-y)^2}$$

$$= \frac{x - y - x - y}{(x-y)^2} = \frac{-2y}{(x-y)^2}$$

$$\frac{\partial v}{\partial y} = \frac{(x-y)(1) - (x+y)(-1)}{(x-y)^2}$$

$$= \frac{x - y + x + y}{(x-y)^2} = \frac{2x}{(x-y)^2}$$

$$J = \begin{vmatrix} \cancel{-\frac{x}{y^2}}\ \frac{1}{y} & -\frac{x}{y^2} \\ \frac{-2y}{(x-y)^2} & \frac{2x}{(x-y)^2} \end{vmatrix}$$

$$= \frac{2}{(x-y)^2 y^2} \begin{vmatrix} y & -x \\ -y & x \end{vmatrix}$$

$$= \frac{2yx}{(x-y)^2 y^2} \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix}$$

$$= \frac{2x}{(x^2-2xy+y^2)y} (1-1)$$

$$= 0$$

$$J = \frac{1}{y}\left(\frac{2x}{(x-y)^2}\right) - \frac{2y}{(x-y)^2}\,\frac{x}{y^2}$$

$$= \frac{2x}{(x-y)^2 y} - \frac{2x}{(x-y)^2 y}$$

$$= 0$$

$\therefore$ u and v are functionally ~~In~~dependent.

$$v = \frac{x+y}{x-y}$$

$$= \frac{y\left(\frac{x}{y}+1\right)}{y\left(\frac{x}{y}-1\right)} =$$

$$v = \frac{u+1}{u-1}$$

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5. Find J, $x = u(1-v)$, $y = uv$

$$J = \frac{\partial(x,y)}{\partial(u,v)}$$

$x = u - uv$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$\frac{\partial x}{\partial u} = (1 - v)$ $\qquad$ $\frac{\partial x}{\partial v} = 0 - u = -u$

$\frac{\partial y}{\partial u} = v$ $\qquad$ $\frac{\partial y}{\partial v} = u$

$$J = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$$

$$= u(1-v) + uv$$

$$= u - uv + uv$$

$$J = u$$

6. Find J, $u = x^2$, $v = y^2$

$$J = \frac{\partial(u,v)}{\partial(x,y)}$$

$$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$\frac{\partial u}{\partial x} = 2x$ $\qquad$ $\frac{\partial u}{\partial y} = 0$

$\frac{\partial v}{\partial x} = 0$ $\qquad$ $\frac{\partial v}{\partial y} = 2y$

$$J = \begin{vmatrix} 2x & 0 \\ 0 & 2y \end{vmatrix}$$

$$J = 4xy.$$

7. Find J, $u = \frac{y^2}{x}$, $v = \frac{x^2}{y}$

$$J = \frac{\partial(u, v)}{\partial(x, y)}$$

$$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$\frac{\partial u}{\partial x} = \frac{-y^2}{x^2} \qquad \frac{\partial u}{\partial y} = \frac{2y}{x}.$$

$$\frac{\partial v}{\partial x} = \frac{2x}{y} \qquad \frac{\partial v}{\partial y} = \frac{-x^2}{y^2}$$

$$J = \begin{vmatrix} \frac{-y^2}{x^2} & \frac{2y}{x} \\ \frac{2x}{y} & \frac{-x^2}{y^2} \end{vmatrix}$$

$$= \frac{x^2 y^2}{x^2 y^2} - \frac{2y \times 2x}{xy}$$

$$= 1 - 4$$

$$= -3$$

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8. Find J, $x = u(1+v)$, $y = v(1+u)$

$$J = \frac{\partial(x,y)}{\partial(u,v)}$$

$x = u + uv$

$y = v + uv$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$\frac{\partial x}{\partial u} = 1 + v \qquad \frac{\partial x}{\partial v} = u$$

$$\frac{\partial y}{\partial u} = v \qquad \frac{\partial y}{\partial v} = 1 + u$$

$$J = \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix}$$

$$= (1+v)(1+u) - uv$$

$$= 1 + u + v + uv - uv$$

$$= 1 + u + v$$