UNIT 5–Multiple Integrals

Applications of multiple integrals are to find areas and volume of various bodies just by taking a little part of them into consideration.

In probability theory, it is used to evaluate Probabilities of two dimensional continuous random Variables

Double Integration (Cartesian coordinate)

A double integral is computed by repeated single variable integration; integrate w.r.t one variable treating the other variable as constant.

$$I = \iint\limits_{R} f(x,y)\,dx\,dy \quad \text{where R is the Region.}$$

Formulae!

1. $\int x^n\,dx = \dfrac{x^{n+1}}{n+1}$

2. $\int k\,dx = kx$

3. $\int \dfrac{1}{x}\,dx = \log x$

4. $\int e^x\,dx = e^x$

5. $\int \sin x\,dx = -\cos x$

6. $\int \cos x\,dx = \sin x$

7. $\int \tan x\,dx = \log(\sec x)$ (or) $-\log(\cos x)$

8. $\int \cot x\,dx = \log(\sin x) = \log(\csc x)$

9. $\int \csc x\,dx = \log[\csc x - \cot x]$

10. $\int \sec x\,dx = \log[\sec x + \tan x]$

11. $\int \sec^2 x\,dx = \tan x$

12. $\int \csc^2 x\,dx = -\cot x$.

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13) $\int \sec x \tan x \, dx = \sec x$

14. $\int \csc x \cot x \, dx = -\csc x$

15. $\int \frac{da}{x^2+a^2} = \frac{1}{a} \tan^{-1}\left[\frac{x}{a}\right]$

16. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \log\left(\frac{x-a}{x+a}\right)$

17. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$

18. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x$

19. $\int \frac{-1}{\sqrt{1-x^2}} \, dx = \cos^{-1} x$

20. $\int \frac{1}{1+x^2} dx = \tan^{-1} x$

21. $\int \frac{-1}{1+a^2} dx = \cot^{-1} x$

22. $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x$

23. $\int \frac{-1}{x\sqrt{x^2-1}} dx = \csc^{-1} x$

24. $\int \sqrt{a^2-x^2} \, dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2} \sin^{-1}\frac{x}{a}$

25. $\int \frac{dx}{\sqrt{a^2+x^2}} = \log\left[x+\sqrt{x^2+a^2}\right]$

26. $\int u \, dv = uv - \int v \, du$ 27. Bernoulli's formula. $\int u \vartheta \, dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$

Problems on double integration in cartesian coordinates

I 1. Evaluate $\int_0^1 \int_1^2 x(x+y) \, dy \, dx$

Soln:-

Now $\int_0^1 \int_1^2 x(x+y) \, dy \, dx = \int_0^1 \left[\int_1^2 (x^2 + xy) \, dy \right] dx$

$= \int_0^1 \left[x^2 y + x\frac{y^2}{2} \right]_{y=1}^{2} dx$

$= \int_0^1 \left[\left(2x^2 + \frac{4x}{2}\right) - \left(x^2 + \frac{x}{2}\right)\right] dx$

$= \int_0^1 \left[2x^2 + 2x - x^2 - \frac{x}{2} \right] dx$

$= \int_0^1 \left[x^2 + \frac{3}{2}x \right] dx$

$= \left[\frac{x^3}{3} + \frac{3}{2}\frac{x^2}{2} \right]_{x=0}^{1} = \left[\left(\frac{1}{3} + \frac{3}{4}\right) - 0 \right]$

$\therefore \int_0^1 \int_1^2 x(x+y) \, dy \, dx = \frac{13}{12}$

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2. Evaluate $\int\limits_0^a \int\limits_0^b xy(x-y)\,dx\,dy$

Soln:-

$$\int\limits_0^a \int\limits_0^b xy(x-y)\,dx\,dy = \int\limits_0^a \int\limits_0^b [x^2y - xy^2]\,dx\,dy$$

$$= \int\limits_0^a \left[\frac{x^3y}{3} - \frac{x^2y^2}{2}\right]_{x=0}^b dy$$

$$= \int\limits_0^a \left[\frac{b^3y}{3} - \frac{b^2y^2}{2}\right] dy$$

$$= \left[\frac{b^3}{3}\left(\frac{y^2}{2}\right) - \frac{b^2}{2}\left(\frac{y^3}{3}\right)\right]_{y=0}^a$$

$$= \left[\frac{b^3a^2}{6} - \frac{b^2a^3}{6}\right] - 0$$

$$\int\limits_0^a \int\limits_0^b xy(x-y)\,dx\,dy = \frac{a^2b^2}{6}[b-a]$$

8. Evaluate $\int\limits_2^3 \int\limits_1^2 \frac{1}{xy}\,dx\,dy$

Soln:-

Now $$\int\limits_2^3 \int\limits_1^2 \frac{1}{xy}\,dx\,dy = \int\limits_2^3 \int\limits_1^2 \frac{1}{x}\cdot\frac{1}{y}\,dx\,dy$$

$$= \int\limits_2^3 \frac{1}{y}[\log x]_{x=1}^2 \, dy$$

$$= \int\limits_2^3 \frac{1}{y}[\log 2 - \log 1]\, dy$$

$$= \int\limits_2^3 \frac{1}{y}\log 2 \, dy \qquad [\because \log 1 = 0$$

$$= \log 2 \int\limits_2^3 \frac{dy}{y} = \log 2 \, [\log y]_2^3$$

$$\int\limits_2^3 \int\limits_1^2 \frac{1}{xy}\,dx\,dy = \log 2 \log\left(\frac{3}{2}\right) = \log 2[\log 3 - \log 2] = \log 2 \log\left(\frac{3}{2}\right)$$

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4. Evaluate $\int_0^3 \int_0^2 e^{x+y} \, dy \, dx$.

$$\int_0^3 \int_0^2 e^{x+y} \, dy \, dx = \int_0^3 \int_0^2 e^x \cdot e^y \, dy \, dx = \int_0^3 e^x \left[e^y \right]_{y=0}^2 dx$$

$$= \int_0^3 e^x \left[e^2 - e^0 \right] dx = \left[e^2 - 1 \right] \left[e^x \right]_{x=0}^3$$

$$= \left[e^2 - 1 \right] \left[e^3 - e^0 \right] = \left[e^2 - 1 \right] \left[e^3 - 1 \right]$$

$$\therefore \int_0^3 \int_0^2 e^{x+y} \, dy \, dx = \left[e^2 - 1 \right] \left[e^3 - 1 \right]$$

5. Evaluate $\int_0^5 \int_0^{x^2} x (x^2 + y^2) \, dx \, dy$

Soln:-
$$\int_0^5 \int_0^{x^2} x(x^2 + y^2) \, dx \, dy = \int_0^5 \int_0^{x^2} \left[x^3 + xy^2 \right] dy \, dx$$

$$= \int_0^5 \left[x^3 y + x \frac{y^3}{3} \right]_{y=0}^{x^2} dx$$

$$= \int_0^5 \left[\left(x^5 + \frac{x^7}{3} \right) - (0+0) \right] dx$$

$$= \int_0^5 \left[x^5 + \frac{x^7}{3} \right] dx$$

$$= \left[\frac{x^6}{6} + \frac{x^8}{24} \right]_0^5$$

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$$= \left(\frac{5^6}{6} + \frac{5^8}{24} \right) - 0$$

$$= 5^6 \left[\frac{29}{24} \right]$$

$$\int_0^5 \int_0^{x^2} x(x^2 + y^2) \, dx \, dy = 5^6 \left[\frac{29}{24} \right]$$

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6. Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} dy\,dx$

Soln:-

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} dy\,dx = \int_0^a [y]_{y=0}^{\sqrt{a^2-x^2}}\,dx$$

$$= \int_0^a \sqrt{a^2-x^2}\,dx$$

$$= \left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= \left(0 + \frac{a^2}{2}\sin^{-1}(1)\right) - (0+0)$$

$$= \frac{a^2}{2}\left(\frac{\pi}{2}\right) = \frac{\pi a^2}{4}.$$

7. Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dx\,dy}{1+x^2+y^2}$

Soln:-

$$\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dx\,dy}{1+x^2+y^2} = \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy}{1+x^2+y^2}\,dx$$

$$= \int_0^1 \int_0^1 \frac{dy}{y^2+\left(\sqrt{1+x^2}\right)^2}\,dx$$

$$= \int_0^1 \left(\frac{1}{\sqrt{1+x^2}}\tan^{-1}\left[\frac{y}{\sqrt{1+x^2}}\right]\right)_{y=0}^{\sqrt{1+x^2}}\,dx$$

$$= \int_0^1 \left[\frac{1}{\sqrt{1+x^2}}\tan^{-1}(1) - 0\right]dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}}\left(\frac{\pi}{4}\right)dx = \frac{\pi}{4}\int_0^1 \frac{dx}{\sqrt{1+x^2}}$$

$$= \frac{\pi}{4}\left[\log\left(x+\sqrt{x^2+1^2}\right)\right]_0^1$$

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$$= \frac{\pi}{4}\left[\log(1+\sqrt{2}) - \log(0+1)\right]$$

$$= \frac{\pi}{4}\log(1+\sqrt{2})$$

8. Evaluate $\displaystyle\int_0^a \int_0^{\sqrt{ay}} xy\, dx\, dy$

Soln:-

$$\int_0^a \int_0^{\sqrt{ay}} xy\, dx\, dy = \int_0^a \left[\frac{x^2}{2}y\right]_{x=0}^{\sqrt{ay}} dy$$

$$= \int_0^a \left[\frac{ay}{2}y - 0\right] dy$$

$$= \int_0^a \frac{ay^2}{2}\, dy$$

$$= \frac{a}{2}\left[\frac{y^3}{3}\right]_0^a$$

$$= \frac{a}{2}\left[\frac{a^3}{3}\right]$$

$$= \frac{a^4}{6}$$

$$\int_0^a \int_0^{\sqrt{ay}} xy\, dx\, dy = \frac{a^4}{6}.$$