EULER'S THEOREM :-

HOMOGENEOUS FUNCTION :

A function $f(x,y)$ is said to be homogeneous function of degree 'n'. If $f(tx, ty) = t^n f(x,y)$

EULER'S STATEMENT :

If $f(x,y)$ is homogenous function of degree n. The euler's theorem states that $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$.

PROBLEMS :

1. Prove $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$ by using euler's theorem. where $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$

Soln :-

Given : $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$

$$f(x,y) = \tan u = \frac{x^3+y^3}{x-y}$$

$$f(tx,ty) = \frac{(tx)^3+(ty)^3}{tx-ty} = \frac{t^3x^3+t^3y^3}{tx-ty}$$

$$= \frac{t^3(x^3+y^3)}{t(x-y)}$$

$$= t^2\frac{(x^3+y^3)}{x-y}$$

$$f(tx,ty) = t^2 f(x,y)$$

$$x \cos u \frac{\partial u}{\partial x} + y \cos u \frac{\partial u}{\partial y} = 0.$$

$$\cos u \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}\right) = 0.$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0.$$

Hence proved.

3. Verify Euler's theorem for $u = x^3 \sin\left(\frac{y}{x}\right)$

$$f(x,y) = u = x^3 \sin\left(\frac{y}{x}\right)$$

$$f(tx, ty) = (tx)^3 \sin\left(\frac{ty}{tx}\right)$$

$$= t^3 x^3 \sin\left(\frac{y}{x}\right)$$

$$= t^3 u$$

$$= t^3 f(x,y)$$

$\therefore$ u is a homogeneous function of degree 3. By Euler's theorem

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu$$

$u = x^3 \quad v = \sin\left(\frac{y}{x}\right)$

$u' = 3x^2; \quad v' = \cos\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[x^3 \sin\left(\frac{y}{x}\right)\right]$$

$$= 3x^2 \sin\left(\frac{y}{x}\right) + x^3 \cos\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)$$

$$= 3x^2 \sin\left(\frac{y}{x}\right) + x^3 \left(\frac{-y}{x^2}\right) \cos\left(\frac{y}{x}\right)$$

$$= 3x^2 \sin\left(\frac{y}{x}\right) - xy \cos\left(\frac{y}{x}\right)$$

$$x\frac{\partial u}{\partial x} = 3x^3\sin\left(\frac{y}{x}\right) - x^2 y\cos\left(\frac{y}{x}\right) \quad \text{--- (1)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(x^3\sin\left(\frac{y}{x}\right)\right)$$

$$= x^3\cos\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)$$

$$= x^3\left(\frac{1}{x}\right)\cos\left(\frac{y}{x}\right)$$

$$\frac{\partial u}{\partial y} = x^2\cos\left(\frac{y}{x}\right)$$

$$y\frac{\partial u}{\partial y} = x^2 y\cos\left(\frac{y}{x}\right) \quad \text{--- (2)}$$

Adding (1) + (2)

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3x^3\sin\left(\frac{y}{x}\right) - x^2y\cos\left(\frac{y}{x}\right) + x^2y\cos\left(\frac{y}{x}\right)$$

$$= 3x^3\sin\left(\frac{y}{x}\right)$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3u.$$

Hence proved.