Jacobians :

* If u and v are functions of two independent variables of x and y Then the following determinate $\left| \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right|$ is said to be jacobians of u and v with respect to x and y. It is denoted by

$\frac{\partial(u,v)}{\partial(x,y)}$ (or) $\left[\frac{(u,v)}{(x,y)}\right]$ (or) J.

Properties:

* The Jacobian of u, v with respect to x, y is $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

* If u and v are functions of x and y then $\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1$

* If u, v, w are functionaly dependent functions of three independent variables x, y, z then $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$ (or) $\frac{\partial(u,v)}{\partial(x,y)} = 0$.

* $\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)}$.

* Jacobians of u, v, w with respect to x, y, z is

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

(*) very important

Example: 1

Find the $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ if $x = \frac{u^2}{v}$, $y = \frac{v^2}{w}$, $z = \frac{w^2}{u}$.

Given, $\frac{\partial(x,y,z)}{\partial(u,v,w)}$

u^2v^{-1}

$= u^2(-v^{-2})$

$= \frac{-u^2}{v^2}$

Also given,

$x = \frac{u^2}{v}, \quad y = \frac{v^2}{w}, \quad z = \frac{w^2}{u}$

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$x = \frac{u^2}{v}$

$\frac{\partial x}{\partial u} = \frac{2u}{v} \quad \frac{\partial x}{\partial v} = -\frac{u^2}{v^2}$

$\frac{\partial x}{\partial w} = 0.$

$y = \frac{v^2}{w}$

$\frac{\partial y}{\partial u} = 0 \quad \frac{\partial y}{\partial v} = \frac{2v}{w}, \quad \frac{\partial y}{\partial w} = \frac{-v^2}{w^2}$

$z = \frac{w^2}{u}$

$\frac{\partial z}{\partial u} = \frac{-w^2}{u^2} \quad \frac{\partial z}{\partial v} = 0 \quad \frac{\partial z}{\partial w} = \frac{2w}{u}$

$$J = \begin{vmatrix} \frac{2u}{v} & -\frac{u^2}{v^2} & 0 \\ 0 & \frac{2v}{w} & \frac{-v^2}{w^2} \\ \frac{-w^2}{u^2} & 0 & \frac{2w}{u} \end{vmatrix}$$

$$= \frac{2u}{v}\left[\frac{4vw}{vw} - 0\right] + \frac{u^2}{v^2}\left[0 - \frac{v^2w^2}{u^2w^2}\right] + 0$$

$J = 8 - 1 \quad \boxed{J = 7}$

Example : 2

If $u = e^x \cos y$, $v = e^x \sin y$ then find J

Soln:

Given,

$u = e^x \cos y$

$v = e^x \sin y$

$$J = \frac{\partial(u,v)}{\partial(x,y)} \text{ (or) } \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$u = e^x \cos y$

$\frac{\partial u}{\partial x} = e^x \cos y$

$\frac{\partial u}{\partial y} = -e^x \sin y$

$v = e^x \sin y$

$\frac{\partial v}{\partial x} = e^x \sin y$

$\frac{\partial v}{\partial y} = e^x \cos y$

$$J = \begin{vmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{vmatrix}$$

base same power add $e^x \cdot e^x = e^{2x}$

$= e^{2x} \cos^2 y + e^{2x} \sin^2 y$

$= e^{2x} (\cos^2 y + \sin^2 y)$

$= e^{2x} (1)$

$= e^{2x}$

$$\boxed{\therefore J = e^{2x}}$$