Taylor's Series Expansion:

$(x-a)\quad(y-b)$

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right]$$

$$+ \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right]$$

$$+ \frac{1}{3!}\left[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b)f_{xxy}(a,b) + 3(x-a)(x-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)\right] \ldots$$

Example: 1:

Expand $e^x \cos y$ interms of x and y near the origin by taylor's series expansion upto third degree terms

Soln:

Formula:

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right]$$
$$+\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right]$$
$$+\frac{1}{3!}\left[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b)f_{xxy}(a,b) + 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)\right]\ldots$$

$f(x,y) = e^x \cos y$	$f(0,0) = 1$ ($\cos 0 = 1$, $\sin 0 = 0$)
$f_x(x,y) = e^x \cos y$	$f_x(0,0) = 1$
$f_{xx}(x,y) = e^x \cos y$	$f_{xx}(0,0) = 1$
$f_{xxx}(x,y) = e^x \cos y$	$f_{xxx}(0,0) = 1$
$f_{xy}(x,y) = -e^x \sin y$	$f_{xy}(0,0) = 0$
$f_{xxy}(x,y) = -e^x \sin y$	$f_{xxy}(0,0) = 0$
$f_y(x,y) = -e^x \sin y$	$f_y(0,0) = 0$
$f_{yy}(x,y) = -e^x \cos y$	$f_{yy}(0,0) = -1$
$f_{yyy}(x,y) = e^x \sin y$	$f_{yyy}(0,0) = 0$
$f_{xyy}(x,y) = -e^x \cos y$	$f_{xyy}(0,0) = -1$

$$f(x,y) = f(0,0) + \frac{1}{1!}\left[(x-0)f_x(0,0) + (y-0)f_y(0,0)\right]$$
$$+\frac{1}{2!}\left[(x-0)^2 f_{xx}(0,0) + 2(x-0)(y-0)f_{xy}(0,0) + (y-0)^2 f_{yy}(0,0)\right]$$
$$+\frac{1}{3!}\left[(x-0)^3 f_{xxx}(0,0) + 3(x-0)^2(y-0)f_{xxy}(0,0) + 3(x-0)(y-0)^2 f_{xyy}(0,0) + (y-0)^3 f_{yyy}(0,0)\right]\ldots$$
$$= 1 + \frac{1}{1}\left[x(1) + y(0)\right] + \frac{1}{2}\left[x^2(1) + 2(x)(y)(0) + y^2(-1)\right]$$
$$+ \frac{1}{6}\left[x^3(1) + 3(x^2)(y)(0) + 3xy^2(-1) + y^3(0)\right]$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{3xy^2}{6}$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{3xy^2}{6}$$

Example : 3

Find Taylor's series expansion $e^x = \sin y$ near the terms $(-1, \pi/4)$ upto 2° degree

Soln:

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[(x-a) f_x(a,b) + (y-b) f_y(a,b)\right] + \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right]$$

$\sin \pi/4 = \frac{1}{\sqrt{2}}$, $\cos \pi/4 = 1/\sqrt{2}$

$f(x,y) = e^x \sin y$	$f(-1, \pi/4) = \frac{1}{e\sqrt{2}}$
$f_x(x,y) = e^x \sin y$	$f_x(-1, \pi/4) = \frac{1}{e\sqrt{2}}$
$f_{xx}(x,y) = e^x \sin y$	$f_{xx}(-1, \pi/4) = \frac{1}{e\sqrt{2}}$
$f_{xy}(x,y) = e^x \cos y$	$f_{xy}(-1, \pi/4) = \frac{1}{e\sqrt{2}}$
$f_y(x,y) = e^x \cos y$	$f_y(-1, \pi/4) = \frac{1}{e\sqrt{2}}$
$f_{yy}(x,y) = -e^x \sin y$	$f_{yy}(-1, \pi/4) = -\frac{1}{e\sqrt{2}}$

$$f(x,y) = \underset{\frac{1}{e\sqrt{2}}}{f(-1, \pi/4)} + \frac{1}{1!}\left[(x+1) f_x(-1, \pi/4) + (y - \pi/4) f_y(-1, \pi/4)\right]$$

$$+ \frac{1}{2!}\left[(x+1)^2 f_{xx}(-1, \pi/4) + 2(x+1)(y - \pi/4) f_{xy}(-1, \pi/4) + \left(y - \frac{\pi}{4}\right)^2 f_{yy}(-1, \pi/4)\right]$$

$$= \frac{1}{e\sqrt{2}} + \frac{1}{1}\left[(x+1)\frac{1}{e\sqrt{2}} + \left(y - \frac{\pi}{4}\right)\frac{1}{e\sqrt{2}}\right] + \frac{1}{2}\left[(x+1)^2 \frac{1}{e\sqrt{2}} + 2(x+1)(y - \pi/4)\frac{1}{e\sqrt{2}} + (y - \pi/4)^2 (-1/e\sqrt{2})\right]$$

$$= \frac{1}{e\sqrt{2}}\left[1 + (x+1) + \left(y - \frac{\pi}{4}\right) + \frac{(x+1)^2}{2} + (x+1)(y - \pi/4) - \frac{(y - \pi/4)^2}{2}\right]$$