Triple Integral

Evolute Triple Integral $\int_0^4\int_0^1\int_0^1 (x+y+z)\,dz\,dy\,dx$

$\rightarrow$

$$=\int_0^4\int_0^1\left[x\int_0^1 dz + y\int_0^1 dz + \int_0^1 z\,dz\right]dy\,dx$$

$$=\int_0^4\int_0^1\left[(xz)_0^1+(yz)_0^1+\left[\frac{z^2}{2}\right]_0^1\right]dy\,dx$$

$$=\int_0^4\left[\int_0^1 [x+y+1/2)\,dy\right]dx$$

$$=\int_0^4\left[x\int_0^1 dy+\int_0^1 y\,dy+1/2\int_0^1 dy\right]dx$$

$$=\int_0^4\left[(xy)_0^1+\left(\frac{y^2}{2}\right)_0^1+(y/2)_0^1\right]dx$$

$$=\int_0^4[x+1/2+1/2]\,dx$$

$$=\int_0^4(x+1)\,dx$$

$$=\int_0^4 x\,dx+\int_0^4 dx$$

$$= \left(\frac{x^2}{2}\right)_0^4 + (x)_0^4$$

$$= \frac{16}{2} + 4$$

$$= 12$$

Example :- 2

$$\int_0^a \int_0^b \int_0^c (x+y+z)\, dz\, dy\, dx$$

$$\Longrightarrow = \int_0^a \int_0^b \left[x\int_0^c dz + y\int_0^c dz + \int_0^c z\, dz \right] dy\, dx$$

$$= \int_0^a \int_0^b \left[(xz)_0^c + (yz)_0^c + \left(\frac{z^2}{2}\right)_0^c \right] dy\, dx$$

$$= \int_0^a \int_0^b \left[xc) + (yc) + \frac{c^2}{2} \right] dy\, dx$$

$$= \int_0^a \left[xc \int_0^b dy + \int_0^b yc\, dy + \frac{c^2}{2}\int_0^b dy \right] dx$$

$$= \int_0^a \left[\left[xcy \right]_0^b + \left[\frac{y^2 c}{2}\right]_0^b + \left[\frac{yc^2}{2}\right]_0^b \right] dx$$

$$= \int_0^a \left[xbc + \frac{b^2 c}{2} + \frac{bc^2}{2} \right] dx$$

$$= \int_0^a xbc\, dx + \int_0^a \frac{b^2 c}{2}\, dx + \int_0^a \frac{bc^2}{2}\, dx$$

$$= \left[\frac{x^2 bc}{2}\right]_0^a + \left[\frac{b^2 x c}{2}\right]_0^a + \left[\frac{xbc^2}{2}\right]_0^a$$

$$= \frac{a^2 bc}{2} + \frac{ab^2 c}{2} + \frac{abc^2}{2}$$

$$= \frac{abc(abc)}{2}$$

Example: 4

Find the volume of the sphere $x^2+y^2+z^2=a^2$ without transformation.

$x^2+y^2+z^2=a^2$

$z^2=a^2-x^2-y^2$

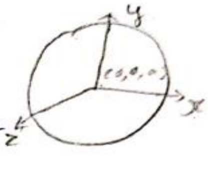
$z=\sqrt{a^2-x^2-y^2}$

→

x varies from $x=0$ to a

y varies from $y=0$ to $y=\sqrt{a^2-x^2}$

z varies from $z=0$ to $z=\sqrt{a^2-x^2-y^2}$



$$\text{Volume } V = 8\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx$$

$z^2=a^2-x^2-y^2$

$z=\sqrt{a^2-x^2-y^2}$

$y=\sqrt{a^2-x^2}$

$$= 8\int_0^a\int_0^{\sqrt{a^2-x^2}}\left[z\right]_0^{\sqrt{a^2-x^2-y^2}} dy\,dx$$

$$= 8\int_0^a\left[\int_0^{\sqrt{a^2-x^2}} \sqrt{(a^2-x^2)-y^2}\,dy\right]dx$$

$a^2=a^2-x^2$

$a=\sqrt{a^2-x^2}$

$$\because \int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$$

Take $a^2=a^2-x^2,\ x^2=y^2$

$a=\sqrt{a^2-x^2},\ x=y$

$$= 8\int_0^a\left[\frac{y}{2}\left(\sqrt{a^2-x^2-y^2}\right)+\left(\frac{a^2-x^2}{2}\right)\sin^{-1}\frac{y}{\sqrt{a^2-x^2}}\right]_0^{\sqrt{a^2-x^2}}dx$$

$$= 8\int_0^a\left[\frac{\sqrt{a^2-x^2}}{2}\sqrt{a^2-x^2-(\sqrt{a^2-x^2})^2}+\frac{a^2-x^2}{2}\sin^{-1}\left[\frac{\sqrt{a^2-x^2}}{\sqrt{a^2-x^2}}\right]\right]dx$$

$\sqrt{(a^2-x^2)-(a^2-x^2)}$ (0)

$$= 8\int_0^a\left(0+\frac{a^2-x^2}{2}\sin^{-1}(1)\right)dx$$

$\pi/2$

$$= 8\int_0^a\left(\frac{a^2-x^2}{2}\left(\frac{\pi}{2}\right)\right)dx$$

$\int\frac{a^2}{2} - \frac{x^2}{2}dx$

$\int_0^a\frac{a^2}{2}dx - \int_0^a\frac{x^2}{2}dx$

$\left(\frac{a^2x}{2}\right)_0^a - \left(\frac{x^3}{6}\right)_0^a$

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$$= 2\pi\left[\int_0^a a^2 dx - \int_0^a x^2 dx\right]$$

$$= 2\pi\left[\left[a^2x\right]_0^a - \left[\frac{x^3}{3}\right]_0^a\right] \qquad = 4\pi\left(\frac{a^3}{2} - \frac{a^3}{6}\right)$$

$$= 2\pi\left[a^3 - \frac{a^3}{3}\right] \qquad = 4\pi\left(\frac{6a^3 - 2a^3}{12}\right)$$

$$= 2\pi\left[\frac{3a^3 - a^3}{3}\right] \qquad \left(\frac{4a^3}{12}\right)4\pi$$

$$= \frac{4}{3}\pi a^3 \qquad 4\pi\left(\frac{a^3}{3}\right) \qquad \frac{4}{3}\pi a^3$$

Example:- 5

Evolute (triple Integral) $\iiint dx\,dy\,dz$ Where V is the region of a space inside the tetrahectant bounded by the planes $x=0$, $y=0$, $z=0$ and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

$\Rightarrow$

$$\therefore \frac{z}{c} = 1 - y/b - x/a$$

x varies from 0 to $x=a$ $\qquad z = c(1 - y/b - x/a)$

y varies from 0 to $y = b[1 - x/a]$

z varies from 0 to $z = c[1 - x/a - y/b]$

$$\text{Volume } V = \int_0^a \int_0^{b[1-x/a]} \left[\int_0^{c[1-x/a-y/b]} dz\right] dy\, dx$$

$$= \int_0^a \int_0^{b[1-x/a]} [z]^{c(1-x/a-y/b)} dy\, dx$$

$$= c\int_0^a \left[\int_0^{b[1-x/a]} (1 - x/a - y/b)\, dy\right] dx$$

$$= c\int_0^a \left[\int_0^{b(1-x/a)} dy - \frac{x}{a}\int_0^{b(1-x/a)} dy - \frac{1}{b}\int_0^{b(1-x/a)} y\, dy\right] dx$$

$$= c\int_0^a \left[[y]_0^{b(1-x/a)} - \left[\frac{x}{a}[y]\right]_0^{b(1-x/a)} - \left[\frac{y^2}{2b}\right]_0^{b(1-x/a)} \right] dx$$

$$= c\int_0^a \left[b(1-x/a) - \frac{bx}{a}\left[1-x/a\right] - \frac{b^2\left[1-\frac{x}{a}\right]^2}{2b} \right] dx$$

$$= bc\int_0^a \left[1 - x/a - x/a + x^2/a^2 - \frac{1 - x^2/a^2 + 2(x/a)}{2} \right] dx$$

$$= bc\int_0^a \left[1 - x/a - x/a + x^2/a^2 - 1/2 - \frac{x^2}{2a^2} + \frac{2x}{2a} \right] dx$$

$$= bc\int_0^a 1/2 - x/a + x^2/a^2 - x^2/2a^2 \; dx$$

$$= bc\left[1/2\int_0^a dx - 1/a\int_0^a x\,dx + 1/a^2\int_0^a x^2dx - 1/2a^2\int_0^a x^2dx \right]$$

$$= bc\left[1/2[x]_0^a - 1/a\left[\frac{x^2}{2}\right]_0^a + 1/a^2\left[\frac{x^3}{3}\right]_0^a - 1/2a^2\left[x^3/3\right]_0^a \right]$$

$$= bc\left[\frac{a}{2} - 1/a\left[\frac{a^2}{2}\right] + 1/a^2\left[\frac{a^3}{3}\right] - \frac{1}{2a^2}\left[\frac{a^3}{3}\right] \right]$$

$$= bc\left[\frac{a}{2} - 1/a\left[\frac{a^2}{2}\right] + 1/a^2\left[\frac{a^3}{3}\right] - 1/2a^2\left[\frac{a^3}{3}\right] \right]$$

$$= bc\left[a/2 - a/2 + a/3 - a/6 \right]$$

$$= bc\left[\frac{6a-3a}{18} \right]$$

$$= bc\left[\frac{3a}{18} \right]$$

$$= \frac{abc}{6}$$