Unit - I

Multiple Integrals

Basic definition and Formula

Integrals:-

The process of finding or solving definite and Indefinite Integrals are called Integral or reciprocal of differentiation is called Integral.

Formulae:-

1. $\int 1\,dx = x+c$
2. $\int a\,dx = ax+c$
3. $\int x^n dx = \frac{x^{n+1}}{n+1}+c, n \neq -1$
4. $\int \sin x\,dx = -\cos x+c$
5. $\int \cos x\,dx = \sin x+c$
6. $\int \sec^2 x\,dx = \tan x+c$
7. $\int \operatorname{cosec}^2 x\,dx = -\cot x+c$
8. $\int \sec x(\tan x)dx = -\sec x+c$
9. $\int \operatorname{cosec} x(\cot x)dx = -\operatorname{cosec} x+c$
10. $\int \frac{1}{x}dx = \log|x|+c$
11. $\int e^x dx = e^x+c$
12. $\int a^x dx = \frac{a^x}{\log a}+c \; a>0, a\neq 1$
13. $\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}x+c$
14. $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}(x/a)$
15. $\int \frac{1}{1+x^2}dx = \tan^{-1}x+c$
16. $\int \frac{1}{|x|\sqrt{x^2-1}}dx = \sec^{-1}x+c$

Double Integral

A double integral ($\iint$) is a way to integrate over a two dimensional area.

Just as an ordinary integral allows you to find the area under the curve, a double helps to find volume under two dimensional area or a surface.

Example:-1

$$\int_0^1 \int_0^2 (x+y)\,dy\,dx$$

$$\Rightarrow \int_0^1 \left[\int_0^2 (x+y)\,dy\right]dx$$

$$= \int_0^1 \left[\int_0^2 (x\,dy + y\,dy)\right]dx$$

$$= \int_0^1 \left[\int_0^2 x\,dy + \int_0^2 y\,dy\right]dx$$

$$= \int_0^1 \left[[xy]_0^2 + \left[\frac{y^2}{2}\right]_0^2 \right]dx$$

$$= \int_0^1 \left[[2x - 0(x)] + \left[\frac{2^2}{2} - \frac{0^2}{2}\right] \right] dx$$

$$= \int_0^1 (2x+2)\,dx$$

$$= \int_0^1 2x\,dx + \int_0^1 2\,dx$$

$$= \left[\frac{2x^2}{2}\right]_0^1 + (2x)_0^1$$

$$= 2/2 + 2$$

$$= 3$$

Example:-2

Solve $\int_2^b \int_2^a \frac{dx\,dy}{xy}$

$\Rightarrow = \int_2^b \left[\int_2^a \frac{dx}{x} \right] \frac{dy}{y}$

$= \int_2^b [\log x]_2^a \frac{dy}{y}$

$= \int_2^b (\log a - \log 2) \frac{dy}{y}$

$= \int_2^b \log a/2 \; dy/y$

$= \log a/2 \int_2^b dy/y$

$= \log a/2 \, [\log y]_2^b$

$= \log a/2 \, [\log b - \log 2]$

$= \log a/2 + \log b/2$

$= \log \frac{a+b}{2}$

Example:- 3

Evaluate $\int_0^1 \int_0^{x^2} (x^2+y^2)\,dy\,dx$

$\Rightarrow = \int_0^1 \left[\int_0^{x^2} (x^2+y^2)\,dy \right] dx$

$= \int_0^1 \left[\int_0^{x^2} x^2\,dy + \int_0^{x^2} y^2\,dy \right] dx$

$= \int_0^1 \left[(x^0 y)_0^{x^2} + \left(\frac{y^3}{3}\right)_0^{x^2} \right] dx$

Example:-6

Evaluate $\int_0^a \int_0^{\sqrt{ay}} xy \, dx \, dy$

$\Rightarrow$

$$\int_0^a \int_0^{\sqrt{ay}} xy \, dx \, dy$$

$$= \int_0^a \left[\int_0^{\sqrt{ay}} xy \, dx \right] dy$$

$$= \int_0^a \left[\left[\frac{x^2 y}{2} \right]_0^{\sqrt{ay}} \right] dy$$

$$= \int_0^a \left[\frac{(\sqrt{ay})^2 y}{2} - 0 \right] dy$$

$$= \int_0^a \left[\frac{ay^2}{2} \right] dy$$

$$= \int_0^a \frac{ay^2}{2} \, dy$$

$$= \left[\frac{ay^3}{2 \times 3} \right]_0^a$$

$$= \frac{aa^3}{6} - 0$$

$$= \frac{a^4}{6}$$