(1) JACOBIAN :-

If $U = f(x, y)$, $V = g(x, y)$ are the two functions in two variables, then the Jacobian of U & V w.r.t x and y is donated by J

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

properties :-

* $J \cdot J' = 1$
* If u, v and w are functionally independent then $J = 0$.

(1) If $U = e^x \cos y$, $V = e^x \sin y$ Find J

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$= 0 - \frac{z}{x}\left(\frac{-2x}{z}\right) + \frac{y}{x}\left(\frac{2x}{y}\right)$

$= 0 + 2 + 2 = 4.$

$J = 4$

4. If $x = r\cos\theta$, $y = r\sin\theta$ Then find the jacobian of x and y w.r.t. r and θ.

solution:

Jacobian of x & y w.r.t. r & θ is

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$x = r\cos\theta \quad \frac{\partial x}{\partial r} = \cos\theta \quad \frac{\partial x}{\partial \theta} = -r\sin\theta$

$y = r\sin\theta \quad \frac{\partial y}{\partial r} = \sin\theta \quad \frac{\partial y}{\partial \theta} = r\cos\theta.$

$$J = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$= r\cos^2\theta + \sin^2\theta\, r$

$= r(\cos^2\theta + \sin^2\theta)$

$J = r.$

5. If $u = x^2+1$, $v = y^2-2$ Then find $\frac{\partial(u,v)}{\partial(x,y)}$

$u = x^2+1 \quad \frac{\partial u}{\partial x} = 2x \quad \frac{\partial u}{\partial y} = 0$

$v = y^2-2 \quad \frac{\partial v}{\partial x} = 0 \quad \frac{\partial v}{\partial y} = 2y$

$$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$u = \frac{yz}{x}$; $\frac{\partial u}{\partial x} = -\frac{yz}{x^2}$ $\frac{\partial u}{\partial y} = \frac{z}{x}$ $\frac{\partial u}{\partial z} = \frac{y}{x}$.

$v = \frac{zx}{y}$; $\frac{\partial v}{\partial x} = \frac{z}{y}$ $\frac{\partial v}{\partial y} = -\frac{zx}{y^2}$ $\frac{\partial v}{\partial z} = \frac{x}{y}$

$w = \frac{xy}{z}$; $\frac{\partial w}{\partial x} = \frac{y}{z}$ $\frac{\partial w}{\partial y} = \frac{x}{z}$ $\frac{\partial w}{\partial z} = -\frac{xy}{z^2}$

$$J = \begin{vmatrix} -\frac{yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & -\frac{zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & -\frac{xy}{z^2} \end{vmatrix}$$

$$= -\frac{yz}{x^2}\left(\frac{\cancel{z}yx^2}{y^2z^2} - \frac{x^2}{yz}\right) - \frac{z}{x}\left(-\frac{xyz}{yz^2} - \frac{xy}{y^2}\right) + \frac{y}{x}\left(\frac{xz}{yz} + \frac{xyz}{y^2z}\right)$$

$$= -\frac{yz}{x^2}\left(\frac{x^2}{y^2} - \frac{x^2}{yz}\right) - \frac{z}{x}\left(-\frac{x}{z} - \frac{x}{z}\right) + \frac{y}{x}\left(\frac{x}{y} + \frac{x}{y}\right)$$

$y = \frac{v^2}{w}$; $\frac{\partial y}{\partial u} = 0$ $\frac{\partial y}{\partial v} = \frac{2v}{w}$ $\frac{\partial y}{\partial w} = \frac{-v^2}{w^2}$

$z = \frac{w^2}{u}$; $\frac{\partial z}{\partial u} = -\frac{w^2}{u^2}$ $\frac{\partial z}{\partial v} = 0$ $\frac{\partial z}{\partial w} = \frac{2w}{u}$

$$J = \begin{vmatrix} \frac{2u}{v} & -\frac{u^2}{v^2} & 0 \\ 0 & \frac{2v}{w} & \frac{-v^2}{w^2} \\ -\frac{w^2}{u^2} & 0 & \frac{2w}{u} \end{vmatrix}$$

$$= \frac{2u}{v}\left(\frac{4vw}{uw} + 0\right) + \frac{u^2}{v^2}\left(0 - \frac{v^2w^2}{u^2w^2}\right) + 0$$

$$= 8 + \frac{u^2}{v^2}\left(-\frac{v^2w^2}{u^2w^2}\right) + 0$$

$$= 8 - 1 = 7.$$

3. If $u = \frac{yz}{x}$; $v = \frac{zx}{y}$; $w = \frac{xy}{z}$ find. Jacobian, J.

$$J = \frac{\partial(u, v, w)}{\partial(x, y, z)}$$

$u = e^x \cos y \qquad v = e^x \sin y$

$\frac{\partial u}{\partial x} = e^x \cos y \qquad \frac{\partial v}{\partial x} = e^x \sin y$

$\frac{\partial u}{\partial y} = -e^x \sin y \qquad \frac{\partial v}{\partial y} = e^x \cos y$

$$J = \begin{vmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{vmatrix}$$

$$= e^{2x}\cos^2 y + e^{2x}\sin^2 y$$

$$= e^{2x}[\cos^2 y + \sin^2 y] = e^{2x}$$

$$J = e^{2x}$$

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2. $x = \frac{u^2}{v}$; $y = \frac{v^2}{w}$; $z = \frac{w^2}{u}$

$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)}$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$x = \frac{u^2}{v}$; $\frac{\partial x}{\partial u} = \frac{2u}{v} \qquad \frac{\partial x}{\partial v} = \frac{-u^2}{v^2} \qquad \frac{\partial x}{\partial w} = 0$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x & 0 \\ 0 & 2x \end{vmatrix}$$

$$J = 4xy.$$

6. Show that $u = \frac{x}{y}$, $v = \frac{x+y}{x-y}$ are functionally dependent. Find the relation between them.

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}.$$

$$U = \frac{x}{y} \qquad \frac{\partial u}{\partial x} = \frac{1}{y} \qquad \frac{\partial u}{\partial y} = \frac{-x}{y^2}$$

$$v = \frac{x+y}{x-y}$$

$$\frac{\partial v}{\partial x} = \frac{(x-y)-(x+y)}{(x-y)^2} = \frac{x-y-x-y}{(x-y)^2} = \frac{-2y}{(x-y)}$$

$$\frac{\partial v}{\partial y} = \frac{-(x+y)-(x-y)}{(x-y)^2} = \frac{-x-y-x+y}{(x-y)^2}$$

$$= \frac{-2x}{(x-y)^2}$$

$$J = \begin{vmatrix} \frac{1}{y} & -x/y^2 \\ \frac{-2y}{(x-y)^2} & \frac{2x}{(x-y)^2} \end{vmatrix}$$

$$J = \begin{vmatrix} \frac{-y^2}{x^2} & \frac{2y}{x} \\ \frac{2x}{y} & -\frac{x^2}{y^2} \end{vmatrix}$$

$$= \left(\frac{-y^2}{x^2}\right)\left(-\frac{x^2}{y^2}\right) - \left(\frac{2x}{y}\right)\left(\frac{2y}{x}\right)$$

$$= 1 - 4$$

$$J = -3$$

10. Find J, $x = u(1+v)$; $y = v(1+v)$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$x = u(1+v)$ $y = v(1+u)$

$x = u + uv$ $y = v + uv$

$\frac{\partial x}{\partial u} = 1 + v$ $\frac{\partial y}{\partial u} = v$

$\frac{\partial x}{\partial v} = u$ $\frac{\partial y}{\partial v} = 1 + v$

$$J = \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix}$$

$$= (1+v)(1+u) - uv$$

$$= 1 + u + v + uv - uv$$

$$J = 1 + u + v$$

$$= \frac{1}{y}\left(\frac{2x}{(x-y)^2}\right) - \left(\frac{x}{y^2}\right)\left(\frac{2y}{(x-y)^2}\right)$$

$$= \frac{2x}{y(x-y)^2} - \frac{2x}{y(x-y)^2} = 0$$

u and v are functionally dependent.

$$v = \frac{x+y}{x-y}$$

$$= \frac{y\left(\frac{x}{y}+1\right)}{y\left(\frac{x}{y}-1\right)}$$

$v = \frac{u+1}{u-1}$ is the relation between u & v

7. Find J, $x = u(1-v)$, $y = uv$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$x = u(1-v)$

$\frac{\partial x}{\partial u} = 1-v$

$\frac{\partial x}{\partial v} = -u$

$y = uv$

$\frac{\partial y}{\partial u} = v$

$\frac{\partial y}{\partial v} = u$

$$J = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$$

$= (1-v)u + uv$

$= u - uv + uv$

$J = u$

8. Find J, $u = x^2$, $v = y^2$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$u = x^2$ $\qquad$ $v = y^2$

$\frac{\partial u}{\partial x} = 2x$ $\qquad$ $\frac{\partial v}{\partial x} = 0$

$\frac{\partial u}{\partial y} = 0$ $\qquad$ $\frac{\partial v}{\partial y} = 2y$

$$J = \begin{vmatrix} 2x & 0 \\ 0 & 2y \end{vmatrix} = 4xy$$

$J = 4xy$

9. Find J, $u = \frac{y^2}{x}$; $v = \frac{x^2}{y}$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$u = \frac{y^2}{x}$

$\frac{\partial y}{\partial x} = \frac{-y^2}{x^2}$ $\qquad$ $\frac{\partial v}{\partial x} = \frac{2x}{y}$

$\frac{\partial u}{\partial y} = \frac{2y}{x}$ $\qquad$ $\frac{\partial v}{\partial y} = \frac{-x^2}{y^2}$