Multivariate Analysis:

Sample Mean & sample covariance Matrix:

Formula:

Mean vector:

$$\boxed{\bar{x} = \frac{1}{n} x^T J, \text{ where } x \in R^{n\times p}; J \in R^{n\times 1}}$$

Sample covariance Matrix:

$$\boxed{S = \frac{1}{n-1} x^T \left[I - \frac{1}{n} J \right] x}$$

where, $\left(I - \frac{1}{n} J \right) \in R^{n\times n}$, $I \in R^{n\times n}$

Note:

1) $I_{(2\times 2)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

2) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} -1 & -1 \\ 0 & 3 \end{bmatrix}$

$$A \times B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 0 & 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1+0 & -1+6 \\ -3+0 & -3+12 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & +5 \\ -3 & 9 \end{bmatrix}$$

Example : 1

Let $X = \begin{bmatrix} 4 & 1 \\ -1 & 3 \\ 3 & 5 \end{bmatrix}_{(3\times 2)}$, then Find

(i) sample mean vector $\bar{x}$

(ii) Sample covariance matrix S

Solution:

$n = 3$, $p = 2$

(i) $X^T = \begin{bmatrix} 4 & -1 & 3 \\ 1 & 3 & 5 \end{bmatrix}$, $J = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$\bar{x} = \frac{1}{n} X^T J$$

$$= \frac{1}{3} \begin{bmatrix} 4 & -1 & 3 \\ 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 4 -1 +3 \\ 1 +3 +5 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 6 \\ 9 \end{bmatrix}$$

$$= \begin{bmatrix} 6/3 \\ 9/3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\boxed{\bar{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}}$$

(ii) $\left[I - \frac{1}{n}J\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix}$$

$$S = \frac{1}{n-1} X^T \left[I - \frac{1}{n}J\right] X$$

$$= \frac{1}{2}\begin{bmatrix} 4 & -1 & 3 \\ 1 & 3 & 5 \end{bmatrix}\begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ -1 & 3 \\ 3 & 5 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 14 & -2 \\ -2 & 8 \end{bmatrix} = \begin{bmatrix} 14/2 & -2/2 \\ -2/2 & 8/2 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -1 \\ -1 & 4 \end{bmatrix}$$

$$\boxed{S = \begin{bmatrix} 7 & -1 \\ -1 & 4 \end{bmatrix}}$$

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2) Let $X = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 5 & -4 \\ 0 & 6 & 8 \end{bmatrix}$, find (i) sample mean vector $\bar{x}$ (ii) sample covariance matrix S

Soln: $n = 3$, $p = 3$

(i) $X^T = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 5 & 6 \\ 3 & -4 & 8 \end{bmatrix}$, $J = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$x^T = \frac{1}{n} X^T J = \frac{1}{3}\begin{bmatrix} 1 & 2 & 0 \\ -1 & 5 & 6 \\ 3 & -4 & 8 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 1+2+0 \\ -1+5+6 \\ 3-4+8 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 \\ 10 \\ 7 \end{bmatrix} = \begin{bmatrix} 3/3 \\ 10/3 \\ 7/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 10/3 \\ 7/3 \end{bmatrix}$$

$$\bar{x} = \begin{bmatrix} 1 \\ 10/3 \\ 7/3 \end{bmatrix}.$$

(ii) $$\left[I - \frac{1}{n} J\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix}$$

$$S = \frac{1}{n-1} x^T \left[I + \frac{1}{n} J\right] x$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 2 & 0 \\ -1 & 5 & 6 \\ 3 & -4 & 8 \end{bmatrix}\begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & -1 & -12 \\ -1 & 28.667 & 1.667 \\ -12 & 1.667 & 72.667 \end{bmatrix} \quad \begin{bmatrix} 1 & -1 & 3 \\ 2 & 5 & -4 \\ 0 & 6 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 2/2 & -1/2 & -12/2 \\ -1/2 & \frac{28.667}{2} & 1.667/2 \\ \frac{-12}{2} & \frac{1.667}{2} & \frac{72.667}{2} \end{bmatrix}$$

$$\boxed{S = \begin{bmatrix} 1 & -0.5 & -6 \\ -0.5 & 14.334 & 0.834 \\ -6 & 0.834 & 36.334 \end{bmatrix}}$$