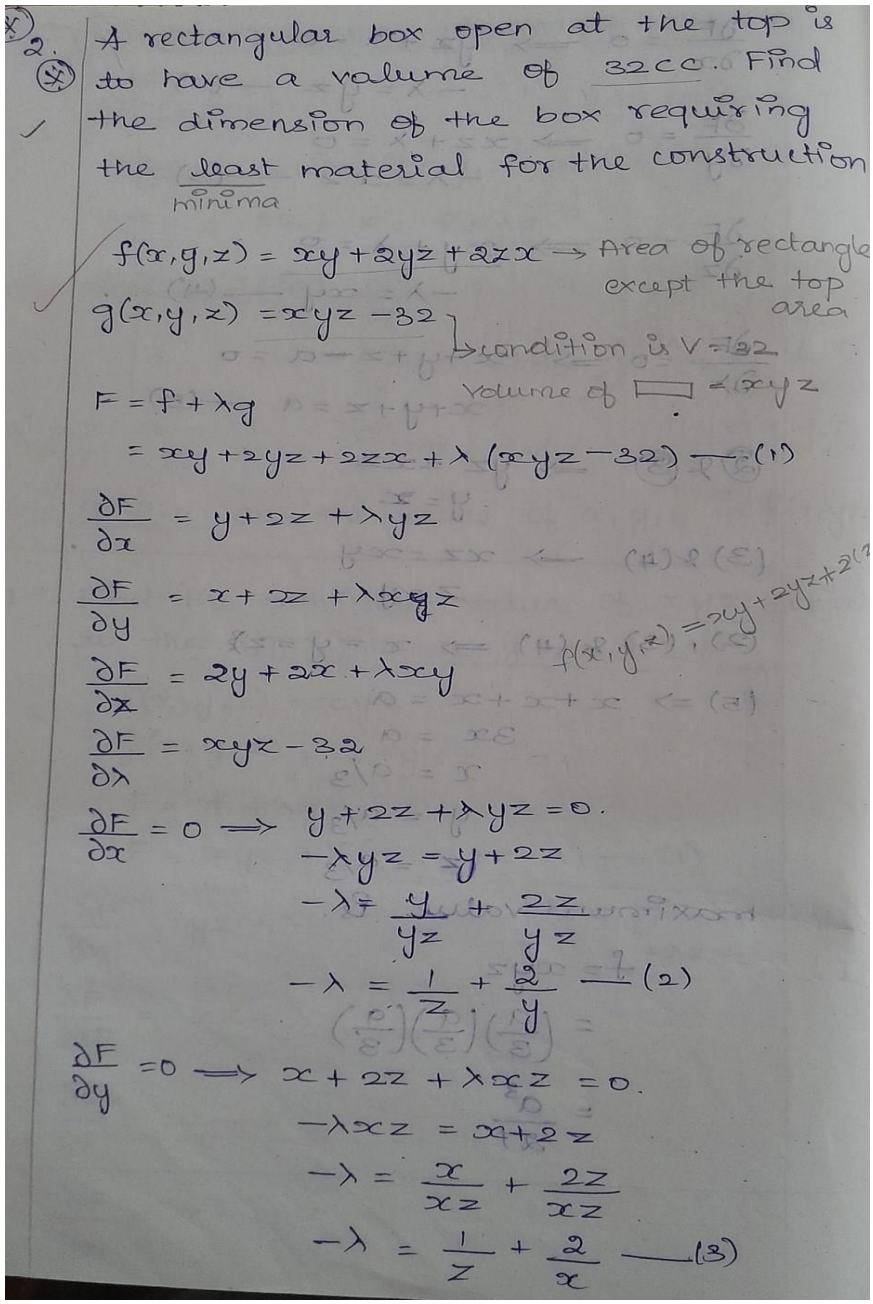
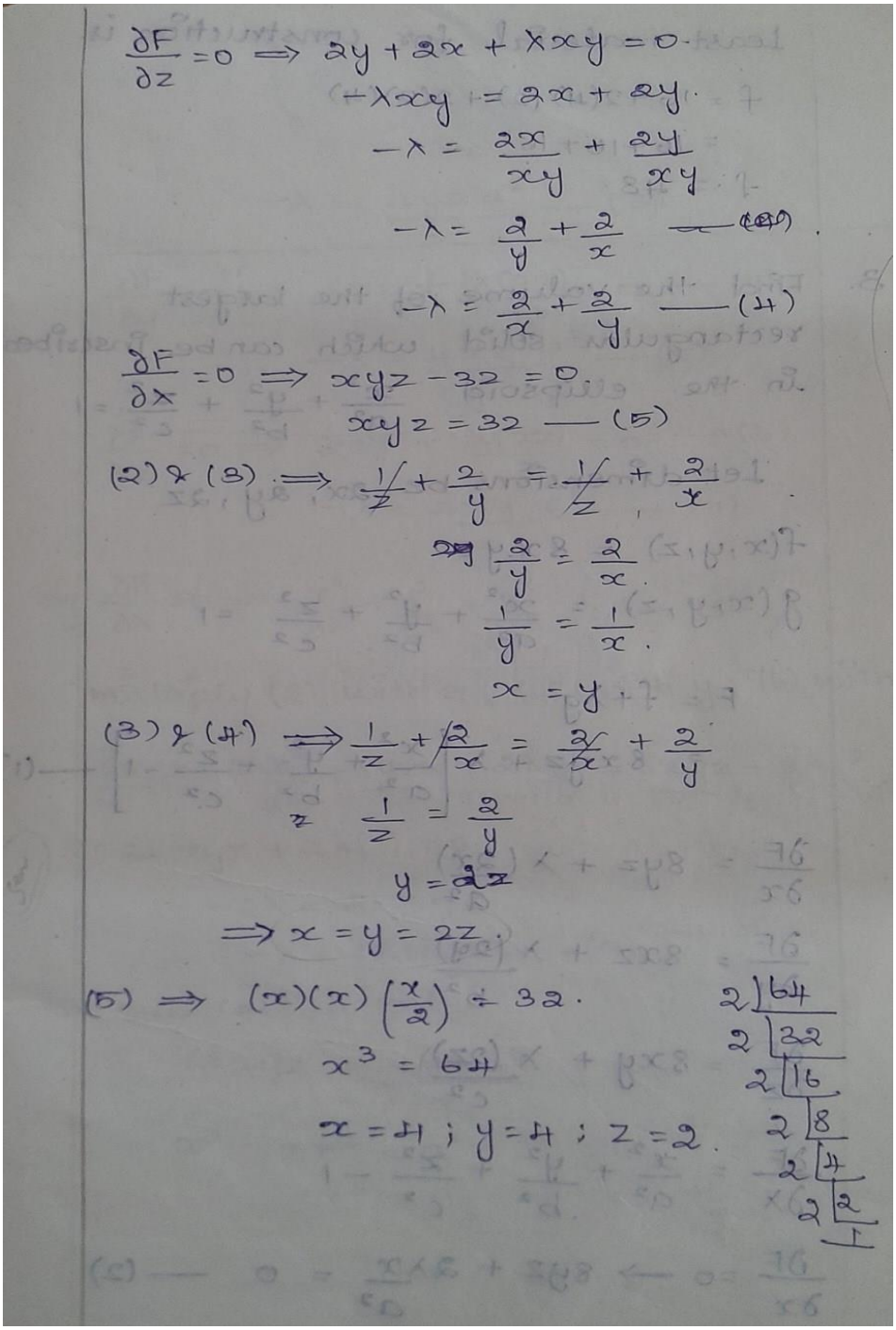
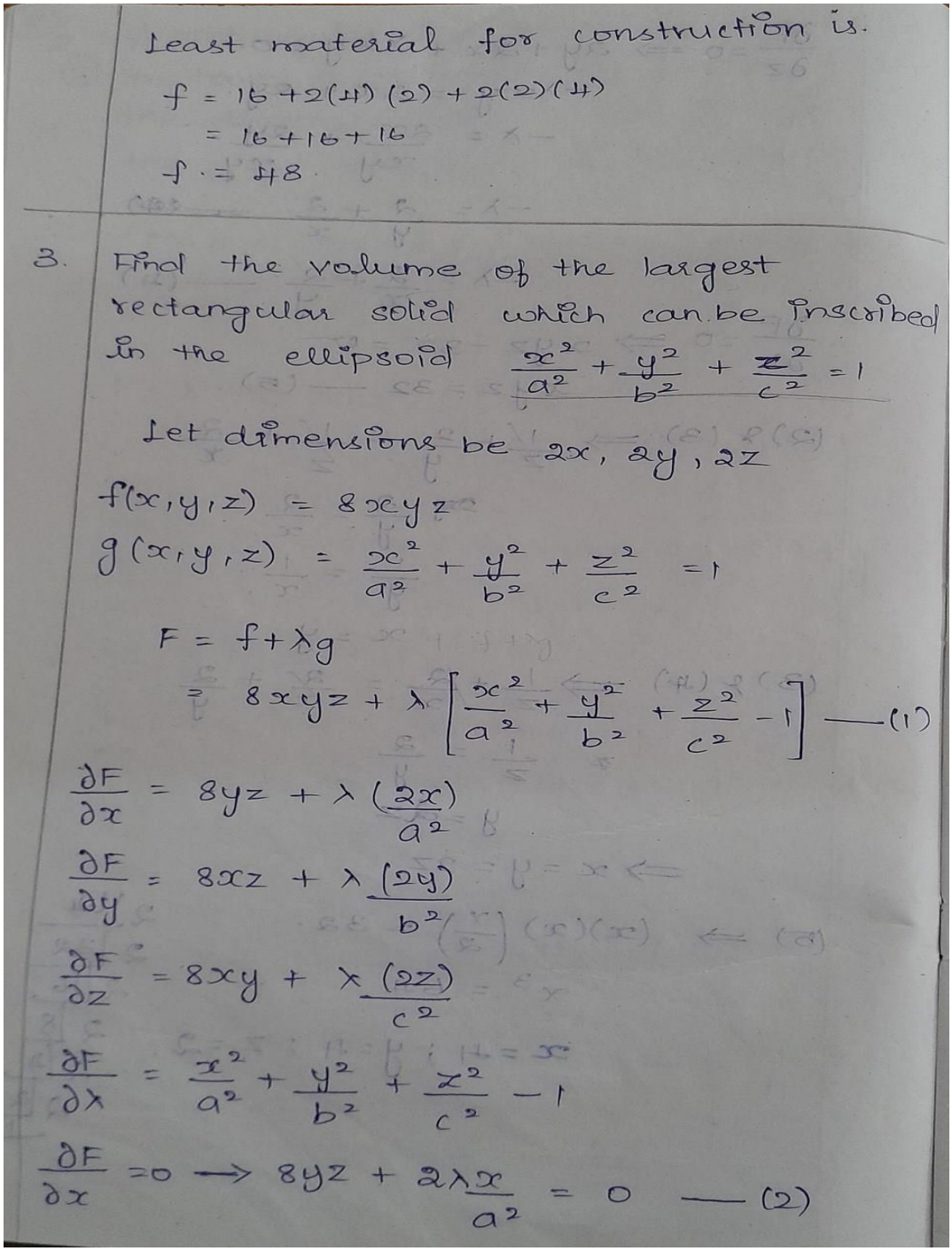
UNIT 4 Lagrange's Method of Underminied Multipliers




$\frac{\partial F}{\partial z} = 0 \Rightarrow 2y + 2x + \lambda xy = 0$
$-\lambda xy = 2x + 2y$
$-\lambda = \frac{2x}{xy} + \frac{2y}{xy}$
$-\lambda = \frac{2}{y} + \frac{2}{x}$ —— (3)
$-\lambda = \frac{2}{x} + \frac{2}{y}$ —— (4)
$\frac{\partial F}{\partial \lambda} = 0 \Rightarrow xyz - 32 = 0$
$xyz = 32$ —— (5)
(2) & (3) $\Rightarrow \frac{1}{z} + \frac{2}{y} = \frac{1}{z} + \frac{2}{x}$
$\frac{2}{y} = \frac{2}{x}$
$\frac{1}{y} = \frac{1}{x}$
$x = y$
(3) & (4) $\Rightarrow \frac{1}{z} + \frac{2}{x} = \frac{2}{x} + \frac{2}{y}$
$\frac{1}{z} = \frac{2}{y}$
$y = 2z$
$\Rightarrow x = y = 2z$
(5) $\Rightarrow (x)(x)\left(\frac{x}{2}\right) = 32$
$x^3 = 64$
$x = 4 ; y = 4 ; z = 2$
2 | 64
2 | 32
2 | 16
2 | 8
2 | 4
2 | 2
1



Least material for construction is.

$f = 16 + 2(4)(2) + 2(2)(4)$

$= 16 + 16 + 16$

$f = 48$

3. Find the volume of the largest rectangular solid which can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Let dimensions be $2x, 2y, 2z$

$f(x,y,z) = 8xyz$

$g(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$F = f + \lambda g$

$= 8xyz + \lambda\left[\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right]$ —— (1)

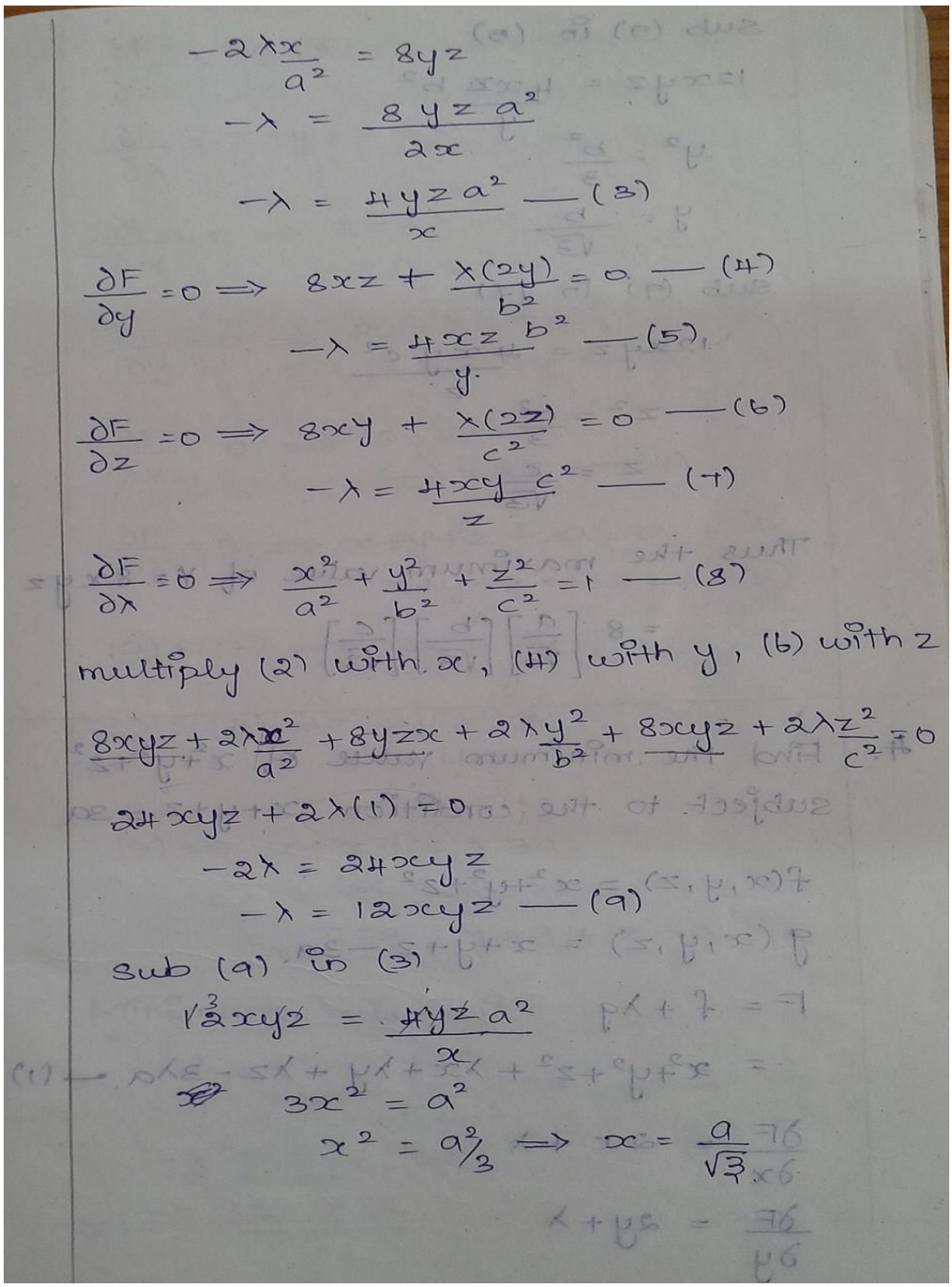
$\frac{\partial F}{\partial x} = 8yz + \lambda\frac{(2x)}{a^2}$

$\frac{\partial F}{\partial y} = 8xz + \lambda\frac{(2y)}{b^2}$

$\frac{\partial F}{\partial z} = 8xy + \lambda\frac{(2z)}{c^2}$

$\frac{\partial F}{\partial \lambda} = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$

$\frac{\partial F}{\partial x} = 0 \longrightarrow 8yz + 2\lambda\frac{x}{a^2} = 0$ —— (2)



$$-2\lambda\frac{x}{a^2} = 8yz$$

$$-\lambda = \frac{8yza^2}{2x}$$

$$-\lambda = \frac{4yza^2}{x} \quad \text{---(3)}$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow 8xz + \frac{\lambda(2y)}{b^2} = 0 \quad \text{---(4)}$$

$$-\lambda = \frac{4xzb^2}{y} \quad \text{---(5)}$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow 8xy + \frac{\lambda(2z)}{c^2} = 0 \quad \text{---(6)}$$

$$-\lambda = \frac{4xyc^2}{z} \quad \text{---(7)}$$

$$\frac{\partial F}{\partial \lambda} = 0 \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{---(8)}$$

multiply (2) with x, (4) with y, (6) with z

$$8xyz + 2\lambda\frac{x^2}{a^2} + 8yzx + 2\lambda\frac{y^2}{b^2} + 8xyz + 2\lambda\frac{z^2}{c^2} = 0$$

$$24xyz + 2\lambda(1) = 0$$

$$-2\lambda = 24xyz$$

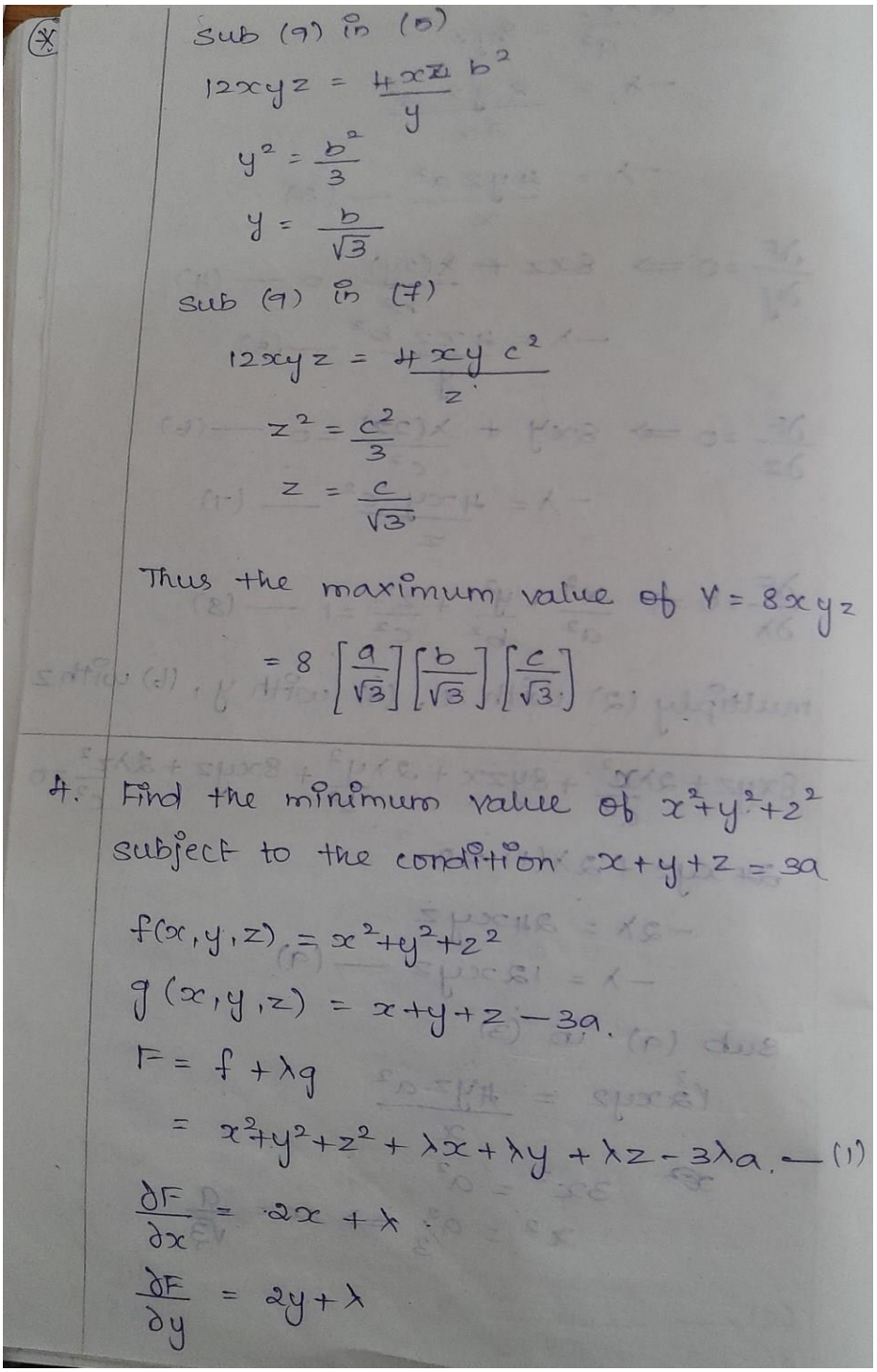
$$-\lambda = 12xyz \quad \text{---(9)}$$

Sub (9) in (3)

$$12xyz = \frac{4yza^2}{x}$$

$$3x^2 = a^2$$

$$x^2 = a^2/3 \Rightarrow x = \frac{a}{\sqrt{3}}$$



Sub (9) in (5)

$$12xyz = \pm\frac{4xz}{y}b^2$$

$$y^2 = \frac{b^2}{3}$$

$$y = \frac{b}{\sqrt{3}}$$

Sub (9) in (7)

$$12xyz = \pm\frac{xy}{z}c^2$$

$$z^2 = \frac{c^2}{3}$$

$$z = \frac{c}{\sqrt{3}}$$

Thus the maximum value of $V = 8xyz$

$$= 8\left[\frac{a}{\sqrt{3}}\right]\left[\frac{b}{\sqrt{3}}\right]\left[\frac{c}{\sqrt{3}}\right]$$

4. Find the minimum value of $x^2+y^2+z^2$ subject to the condition $x+y+z=3a$

$$f(x,y,z) = x^2+y^2+z^2$$

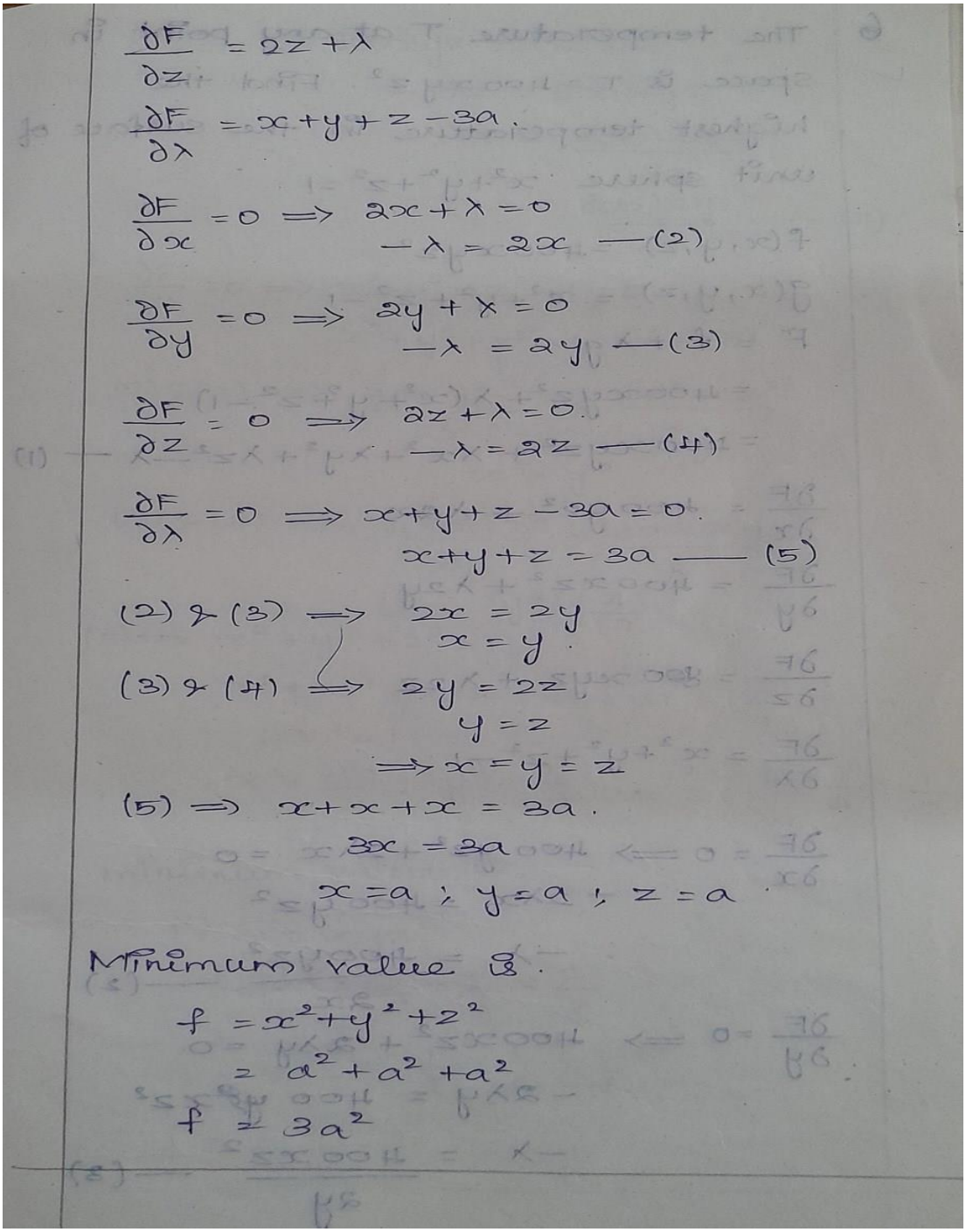
$$g(x,y,z) = x+y+z-3a.$$

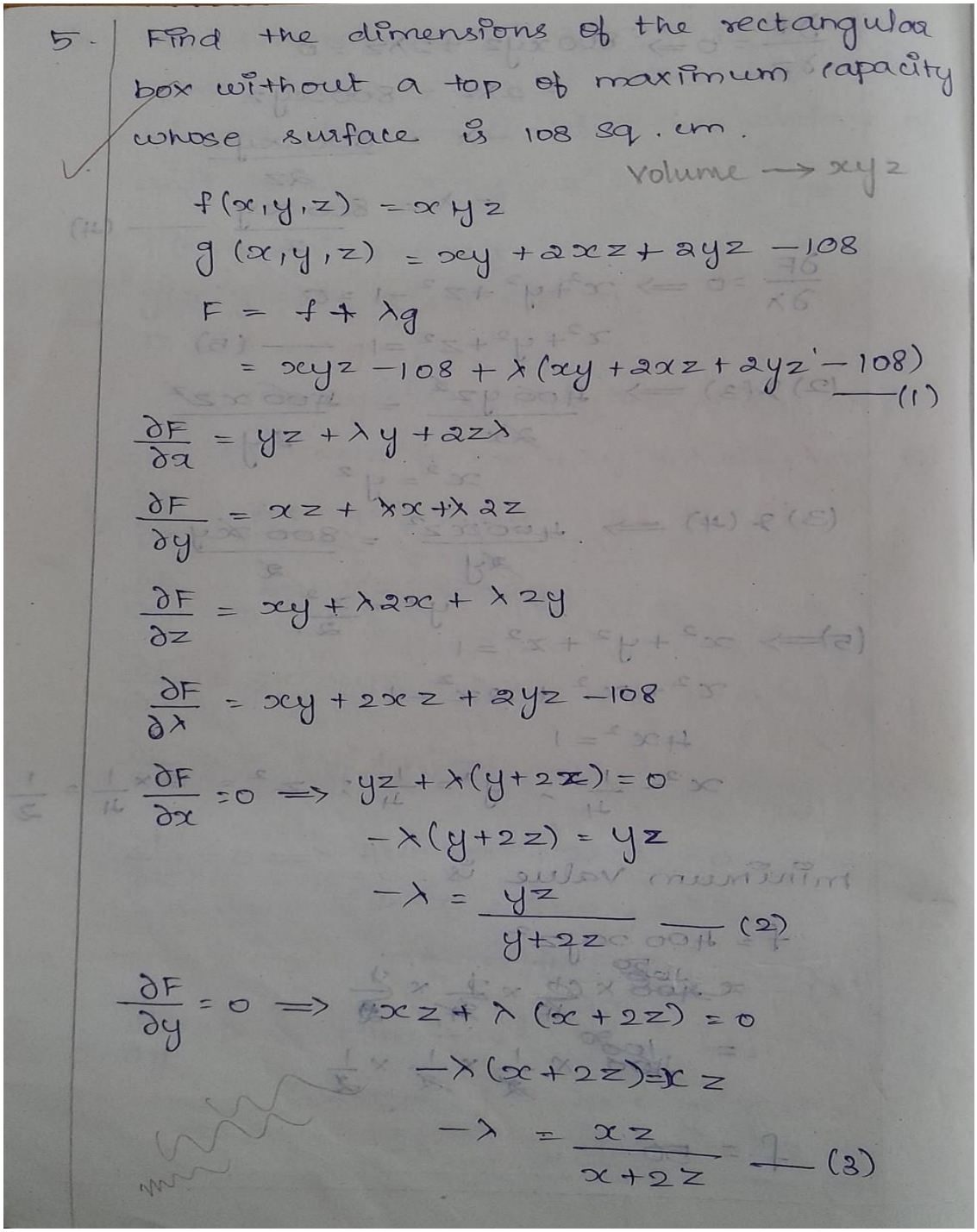
$$F = f+\lambda g$$

$$= x^2+y^2+z^2+\lambda x+\lambda y+\lambda z-3\lambda a. \quad \text{---(1)}$$

$$\frac{\partial F}{\partial x} = 2x+\lambda.$$

$$\frac{\partial F}{\partial y} = 2y+\lambda$$


$\frac{\partial F}{\partial z} = 2z + \lambda$
$\frac{\partial F}{\partial \lambda} = x + y + z - 3a.$
$\frac{\partial F}{\partial x} = 0 \Rightarrow 2x + \lambda = 0$
$-\lambda = 2x \quad (2)$
$\frac{\partial F}{\partial y} = 0 \Rightarrow 2y + \lambda = 0$
$-\lambda = 2y \quad (3)$
$\frac{\partial F}{\partial z} = 0 \Rightarrow 2z + \lambda = 0.$
$-\lambda = 2z \quad (4)$
$\frac{\partial F}{\partial \lambda} = 0 \Rightarrow x + y + z - 3a = 0.$
$x + y + z = 3a \quad (5)$
(2) & (3) $\Rightarrow 2x = 2y$
$x = y.$
(3) & (4) $\Rightarrow 2y = 2z$
$y = z$
$\Rightarrow x = y = z$
(5) $\Rightarrow x + x + x = 3a.$
$3x = 3a$
$x = a ; y = a ; z = a$
Minimum value is.
$f = x^2 + y^2 + z^2$
$= a^2 + a^2 + a^2$
$f = 3a^2$



5. Find the dimensions of the rectangular box without a top of maximum capacity whose surface is 108 sq. cm.

Volume $\rightarrow xyz$

$f(x,y,z) = xyz$

$g(x,y,z) = xy + 2xz + 2yz - 108$

$F = f + \lambda g$

$= xyz - 108 + \lambda(xy + 2xz + 2yz - 108)$ ——(1)

$\frac{\partial F}{\partial x} = yz + \lambda y + 2z\lambda$

$\frac{\partial F}{\partial y} = xz + \lambda x + \lambda 2z$

$\frac{\partial F}{\partial z} = xy + \lambda 2x + \lambda 2y$

$\frac{\partial F}{\partial \lambda} = xy + 2xz + 2yz - 108$

$\frac{\partial F}{\partial x} = 0 \Rightarrow yz + \lambda(y + 2x) = 0$

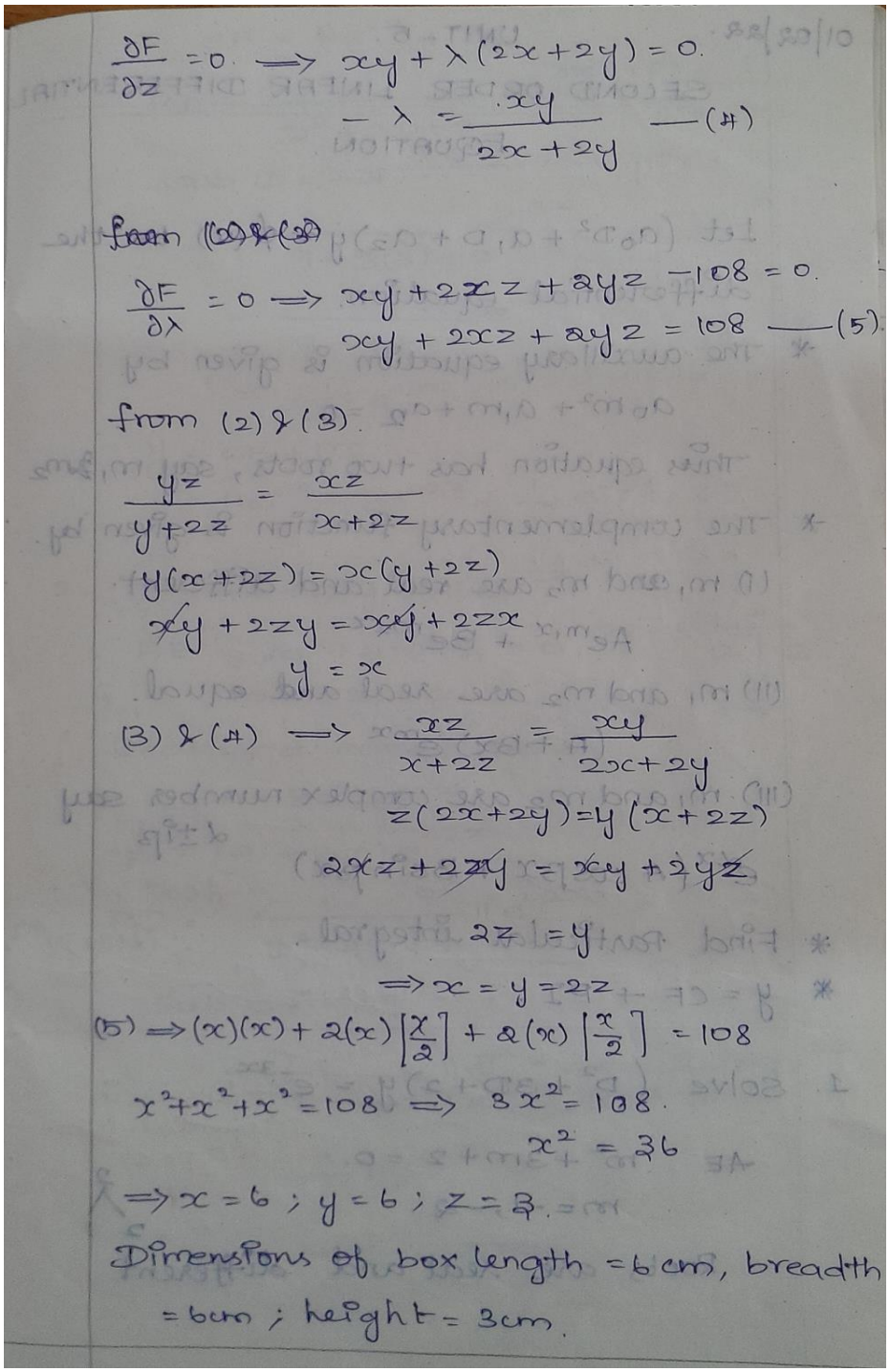
$-\lambda(y + 2z) = yz$

$-\lambda = \frac{yz}{y + 2z}$ ——(2)

$\frac{\partial F}{\partial y} = 0 \Rightarrow xz + \lambda(x + 2z) = 0$

$-\lambda(x + 2z) = xz$

$-\lambda = \frac{xz}{x + 2z}$ ——(3)



$\frac{\partial F}{\partial z} = 0 \Rightarrow xy + \lambda(2x+2y) = 0.$

$-\lambda = \frac{xy}{2x+2y}$ —(4)

$\frac{\partial F}{\partial \lambda} = 0 \Rightarrow xy + 2xz + 2yz - 108 = 0.$

$xy + 2xz + 2yz = 108$ —(5)

from (2) & (3).

$\frac{yz}{y+2z} = \frac{xz}{x+2z}$

$y(x+2z) = x(y+2z)$

$xy + 2zy = xy + 2zx$

$y = x$

(3) & (4) $\Rightarrow \frac{xz}{x+2z} = \frac{xy}{2x+2y}$

$z(2x+2y) = y(x+2z)$

$2xz + 2zy = xy + 2yz$

$2z = y$

$\Rightarrow x = y = 2z$

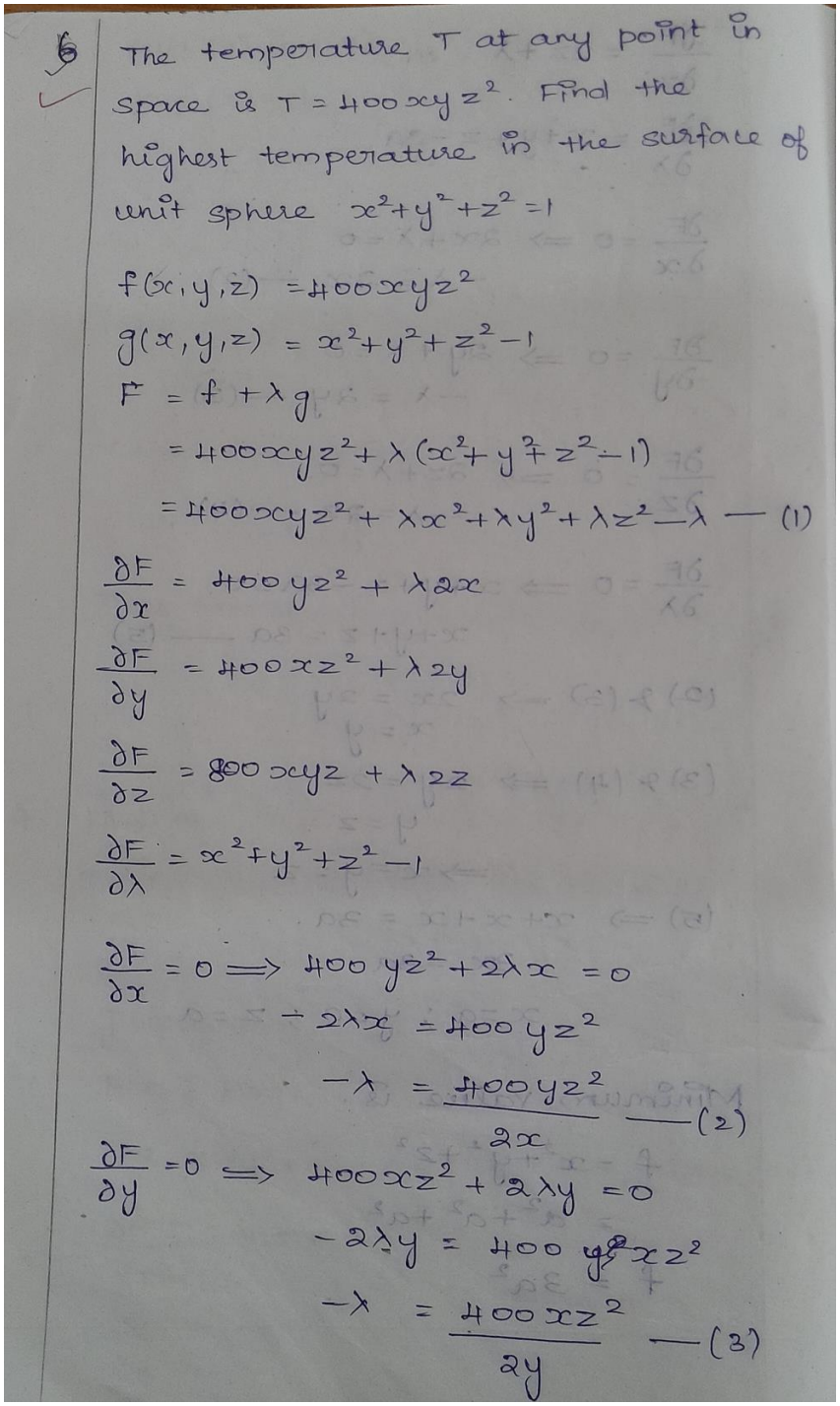
(5) $\Rightarrow (x)(x) + 2(x)\left[\frac{x}{2}\right] + 2(x)\left[\frac{x}{2}\right] = 108$

$x^2 + x^2 + x^2 = 108 \Rightarrow 3x^2 = 108.$

$x^2 = 36$

$\Rightarrow x = 6 ; y = 6 ; z = 3.$

Dimensions of box length = 6 cm, breadth = 6cm ; height = 3cm.



6 The temperature T at any point in space is $T = 400xyz^2$. Find the highest temperature in the surface of unit sphere $x^2+y^2+z^2=1$

$f(x,y,z) = 400xyz^2$

$g(x,y,z) = x^2+y^2+z^2-1$

$F = f + \lambda g$

$= 400xyz^2 + \lambda(x^2+y^2+z^2-1)$

$= 400xyz^2 + \lambda x^2 + \lambda y^2 + \lambda z^2 - \lambda$ — (1)

$\frac{\partial F}{\partial x} = 400yz^2 + \lambda 2x$

$\frac{\partial F}{\partial y} = 400xz^2 + \lambda 2y$

$\frac{\partial F}{\partial z} = 800xyz + \lambda 2z$

$\frac{\partial F}{\partial \lambda} = x^2+y^2+z^2-1$

$\frac{\partial F}{\partial x} = 0 \Rightarrow 400yz^2 + 2\lambda x = 0$

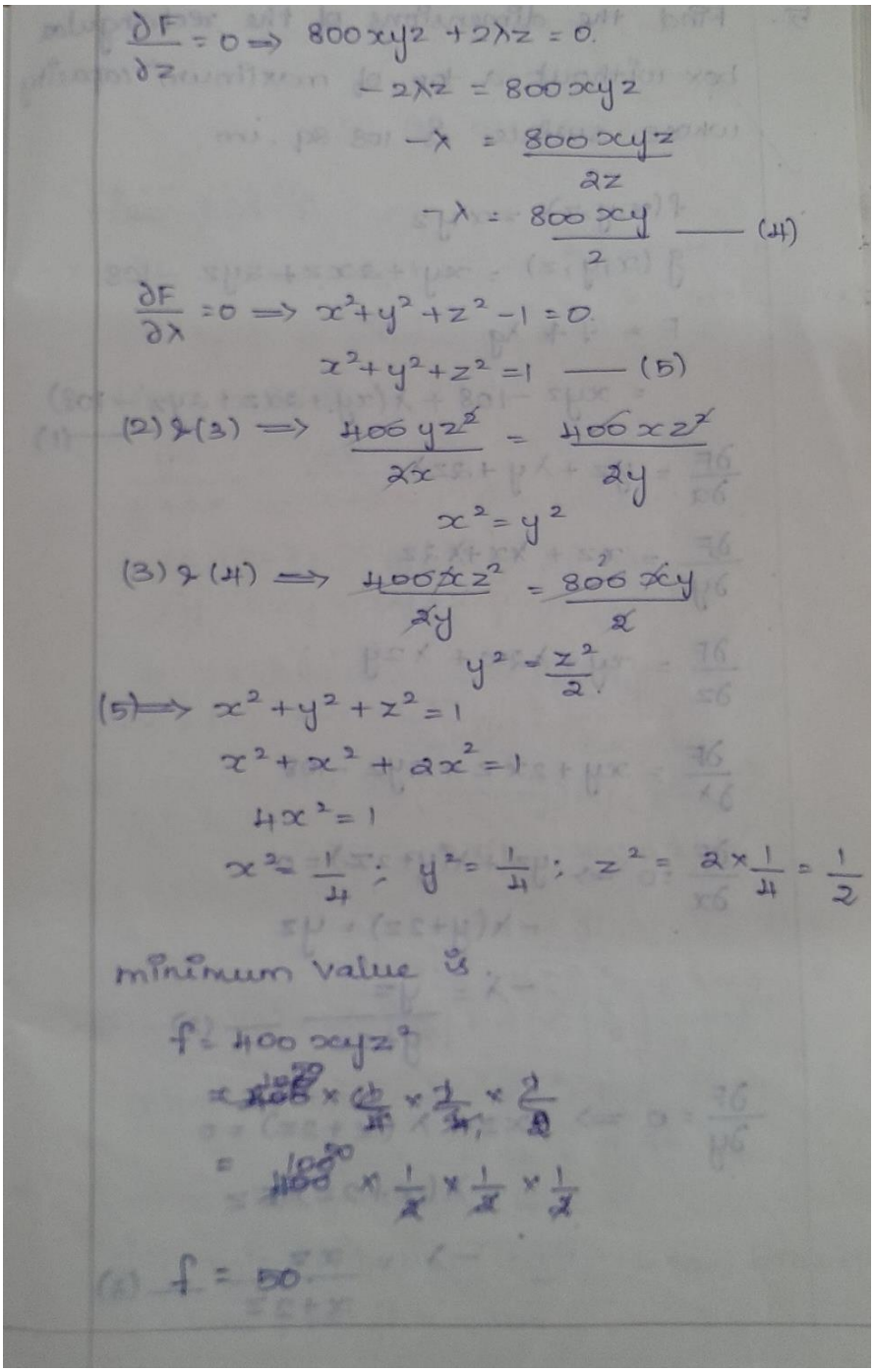
$-2\lambda x = 400yz^2$

$-\lambda = \frac{400yz^2}{2x}$ — (2)

$\frac{\partial F}{\partial y} = 0 \Rightarrow 400xz^2 + 2\lambda y = 0$

$-2\lambda y = 400xz^2$

$-\lambda = \frac{400xz^2}{2y}$ — (3)



$\frac{\partial F}{\partial z} = 0 \Rightarrow 800xyz + 2\lambda z = 0$

$-2\lambda z = 800xyz$

$-\lambda = \frac{800xyz}{2z}$

$-\lambda = \frac{800xy}{2}$ —— (4)

$\frac{\partial F}{\partial \lambda} = 0 \Rightarrow x^2+y^2+z^2-1 = 0$

$x^2+y^2+z^2 = 1$ —— (5)

(2) & (3) $\Rightarrow \frac{400yz^2}{2x} = \frac{400xz^2}{2y}$

$x^2 = y^2$

(3) & (4) $\Rightarrow \frac{400xz^2}{2y} = \frac{800xy}{2}$

$y^2 = \frac{z^2}{2}$

(5) $\Rightarrow x^2+y^2+z^2 = 1$

$x^2+x^2+2x^2 = 1$

$4x^2 = 1$

$x^2 = \frac{1}{4}$; $y^2 = \frac{1}{4}$; $z^2 = 2\times\frac{1}{4} = \frac{1}{2}$

minimum value is

$f = 400xyz^2$

$= 400 \times \frac{1}{2} \times \frac{1}{4} \times \frac{2}{2}$

$= 400 \times \frac{1}{8}$

$f = 50$