Change of order of integration

3) Change of the order of integration and hence evaluate it $\int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} xy\,dy\,dx$.

Sol

The region of integration is bounded by

$x = 0$; $x = 4a$,

$y = \frac{x^2}{4a}$ and $y = 2\sqrt{ax}$

Here, x varies from

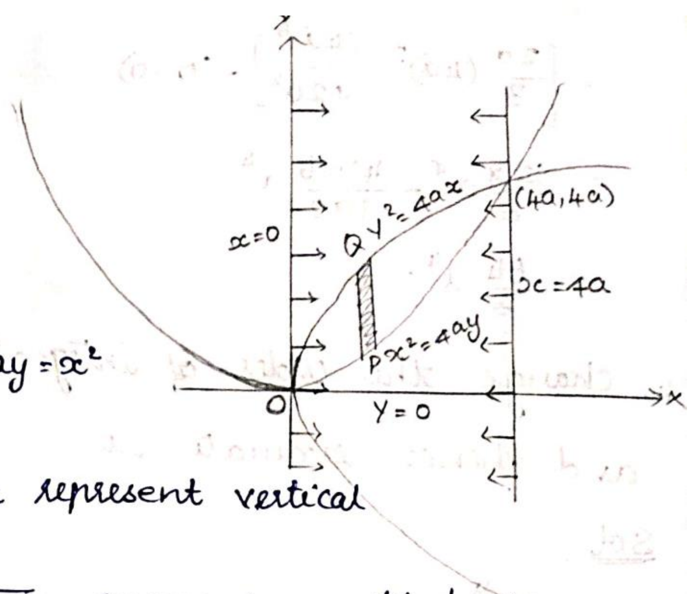
$x = 0$ to $x = 4a$

and y varies from

$y = \frac{x^2}{4a}$ to $y = 2\sqrt{ax}$

(i.e.,) Y varies from $4ay = x^2$

to $y^2 = 4ax$



Here, $x = 0$ to $x = 4a$ represent vertical path and

$y = \frac{x^2}{4a}$ to $y = 2\sqrt{ax}$ represents vertical strip PQ sliding area.

Changing the order of integration is nothing but to change the vertical path into horizontal path and then to change the vertical strip PQ into horizontal strip RS.

Now, $y = 0$ to $y = 4a$ represents horizontal path.

and $x = \frac{y^2}{4a}$ to $x = 2\sqrt{ay}$ represents horizontal strip RS sliding area.

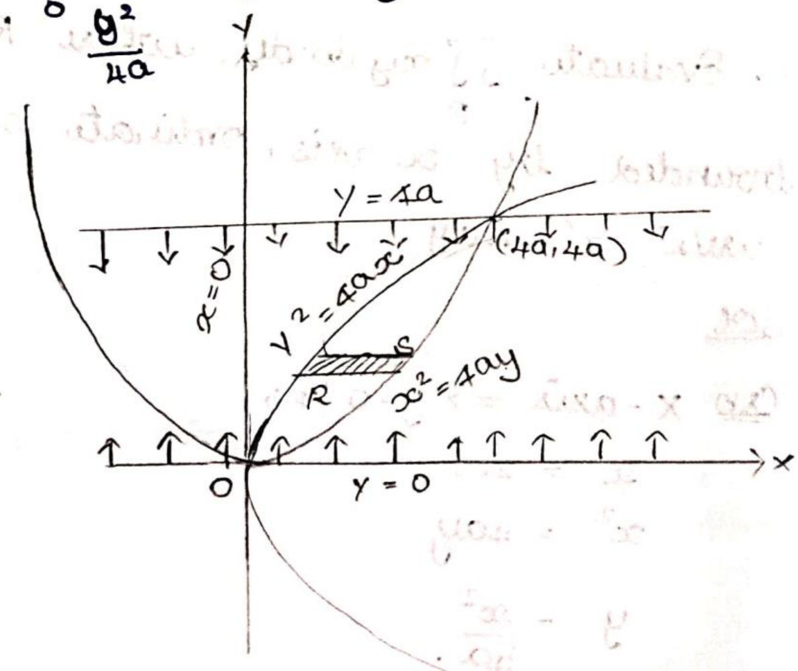
Hence, $\int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} xy\,dy\,dx = \int_0^{4a}\int_{\frac{y^2}{4a}}^{2\sqrt{ay}} xy\,dx\,dy.$

$= \int_0^{4a}\left[\frac{x^2}{2}y\right]_{x=\frac{y^2}{4a}}^{x=2\sqrt{ay}}$

$= \int_0^{4a}\left[\frac{4ay}{2}y - \frac{y^4}{32a^2}y\right]dy$

$= \int_0^{4a}\left[2ay^2 - \frac{y^5}{32a^2}\right]dy$

$= \left[2a\frac{y^3}{3} - \frac{y^6}{192a^2}\right]_0^{4a}$



$$= \left[\frac{2a}{3}(4a)^3 - \frac{(4a)^6}{192a^2}\right] - (0-0)$$

$$= \frac{128}{3}a^4 - \frac{4096}{192}a^4$$

$$= \frac{64}{3}a^4.$$

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4. Change the order of integration $\int_0^\infty \int_0^\infty \frac{e^{-y}}{y}\,dy\,dx$ and hence evaluate it

<u>Sol</u>.

$$= \int_0^\infty \int_0^y \frac{e^{-y}}{y}\,dy\,dx$$

$$= \int_0^\infty \int_0^y \frac{e^{-y}}{y}\,dx\,dy$$

$$= \int_0^\infty \left[\frac{e^{-y}}{y}x\right]_{x=0}^{x=y} dy.$$

$$= \int_0^\infty (e^{-y} - 0)\,dy = \left[\frac{e^{-y}}{-1}\right]_0^\infty$$

$$= -\left[e^{-y}\right]_0^\infty = -[0-1] = 1$$

Now, we divide the region into

$$I = I_1 + I_2$$

$$= \int_0^1 \int_{x^2}^1 xy\,dy\,dx + \int_0^1 \int_1^{2-x} xy\,dy\,dx.$$

In I_1

x varies from $x=0$ to $x=1$

y varies from $y=x^2$ to $y=1$

Here, $x=0$ to $x=1$ represents vertical path and $y=x^2$ to $y=1$ represents vertical strip sliding area.

Changing the order of integration is nothing but to change the vertical path into Horizontal path and to change the vertical strip PQ into horizontal strip RS.

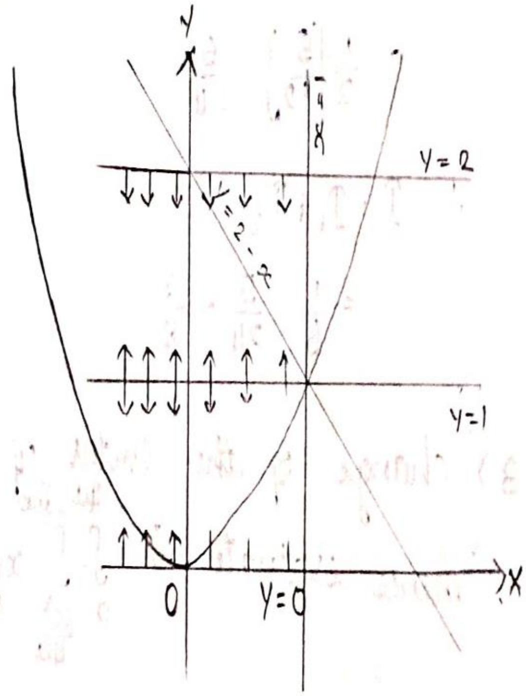
Hence, by changing the order,

we get $\int_0^1 \int_0^{\sqrt{y}} xy\,dx\,dy$.

$= \int_0^1 \left[y\frac{x^2}{2} \right]_{x=0}^{x=\sqrt{y}} dy$.

$= \int_0^1 \left[\frac{y^2}{2} - 0 \right] dy = \frac{1}{2}\int_0^1 y^2 dy$.

$= \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1 = \frac{1}{2}\left[\frac{1}{3}\right] = \frac{1}{6}$.



In I_2

x varies from $x=0$ to $x=1$ and Y axis

$y=1$ to $y=2-x$

Changing the order of integration is nothing but to change the vertical path into horizontal path and to change the vertical strip PQ into horizontal strip RS.

Hence, by changing the order of integration, we get

$$I_2 = \int_1^2 \int_0^{2-y} xy\,dx\,dy = \int_1^2 \left[\frac{x^2}{2} y\right]_{x=0}^{x=2-y} dy$$

$$= \int_1^2 \left[\frac{y(2-y)^2}{2} - 0\right] dy = \frac{1}{2}\int_1^2 (2-y)^2\, y\,dy$$

$$= \frac{1}{2}\int_1^2 (4+y^2-4y)\,y\,dy = \frac{1}{2}\int_1^2 (4y+y^3-4y^2)\,dy$$

$$= \frac{1}{2}\left[4\frac{y^2}{2} + \frac{y^4}{4} - 4\frac{y^3}{3}\right]_1^2$$

$$= \frac{1}{2}\left[2y^2 + \frac{y^4}{4} - \frac{4}{3}y^3\right]_1^2$$

$$= \frac{1}{2}\left[\left(8 + \frac{16}{4} - \frac{32}{3}\right) - \left(2 + \frac{1}{4} - \frac{4}{3}\right)\right]$$

$$= \frac{1}{2}\left[\frac{5}{12}\right] = \frac{5}{24}.$$

$$\therefore\ I = I_1 + I_2$$

$$= \frac{1}{6} + \frac{5}{24} = \frac{3}{8}$$